ON THE JUSTIFICATION OF THE FOLDY-LAX APPROXIMATION FOR THE ACOUSTIC SCATTERING BY SMALL RIGID BODIES OF ARBITRARY SHAPES

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Abstract. We are concerned with the acoustic scattering problem by many small rigid obstacles of arbitrary shapes. We give a sufficient condition on the number $M$ and the diameter $a$ of the obstacles as well as the minimum distance $d$ between them under which the Foldy-Lax approximation is valid. As an application, we study the inverse scattering by the small obstacles in the presence of multiple scattering.

Key words. Acoustic scattering, Small-scatterers, Foldy-Lax approximation, Capacitances, MUSIC algorithm.

AMS subject classifications. 35J08, 35Q61, 45Q05

1. Introduction and statement of the results. Let $B_1, B_2, \ldots, B_M$ be $M$ open, bounded and simply connected sets in $\mathbb{R}^3$ with Lipschitz boundaries containing the origin. We assume that the Lipschitz constants of $B_j$, $j = 1, \ldots, M$ are uniformly bounded. We set $D_m := \epsilon B_m + z_m$ to be the small bodies characterized by the parameter $\epsilon > 0$ and the locations $z_m \in \mathbb{R}^3$, $m = 1, \ldots, M$. Let $U^{i}$ be a solution of the Helmholtz equation $(\Delta + \kappa^2)U^{i} = 0$ in $\mathbb{R}^3$. We denote by $U^{s}$ the acoustic field scattered by the $M$ small bodies $D_m \subset \mathbb{R}^3$ due to the incident field $U^{i}$. We restrict ourselves to (1.) the plane incident waves, $U^{i}(x, \theta) := e^{i k x \cdot \theta}$, with the incident direction $\theta \in S^2$, with $S^2$ being the unit sphere, and (2.) the scattering by rigid bodies. Hence the total field $U^{t} := U^{i} + U^{s}$ satisfies the following exterior Dirichlet problem of the acoustic waves

$$\begin{align*}(\Delta + \kappa^2)U^{t} &= 0 \text{ in } \mathbb{R}^3 \backslash \left( \bigcup_{m=1}^{M} D_m \right),
U^{t}|_{\partial D_m} &= 0, 1 \leq m \leq M,
\frac{\partial U^{s}}{\partial |x|} - i n U^{s} &= o \left( \frac{1}{|x|} \right), |x| \to \infty, \text{ (S.R.C)}
\end{align*}$$

(1.1)

(1.2)

(1.3)

where $\kappa > 0$ is the wave number, $\kappa = 2\pi/\lambda$, $\lambda$ is the wave length and S.R.C stands for the Sommerfeld radiation condition. The scattering problem (1.1-1.3) is well posed in the Hölder or Sobolev spaces, see [9, 10, 18] for instance, and the scattered field $U^{s}(x, \theta)$ has the following asymptotic expansion:

$$U^{s}(x, \theta) = \frac{e^{i k |x|}}{|x|} U^{\infty}(\hat{x}, \theta) + O(|x|^{-2}), \quad |x| \to \infty,$$

(1.4)

with $\hat{x} := \frac{x}{|x|}$, where the function $U^{\infty}(\hat{x}, \theta)$ for $(\hat{x}, \theta) \in S^2 \times S^2$ is called the far-field pattern.

Definition 1.1. We define

1. $a$ as the maximum among the diameters, diam, of the small bodies $D_m$, i.e.

$$a := \max_{1 \leq m \leq M} \text{diam}(D_m) \quad [= \epsilon \max_{1 \leq m \leq M} \text{diam}(B_m)],$$

(1.5)

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1Let us recall that the surface $\partial B_1$ is of Lipschitz class with the Lipschitz constants $r_j, L_j > 0$ if for any $P \in \partial B_1$, there exists a rigid transformation of coordinates under which we have $P = 0$ and $B_j \cap B_1^j(0) = \{ x \in B_1^j(0) : x > \varphi_j(x_1, x_2) \}$, $B_j^2(0)$ being the ball of center 0 and radius $r_j$, where $\varphi_j$ is a Lipschitz continuous function on the disc of center 0 and radius $r_j$, i.e. $B_j^2(0)$, satisfying $\varphi_j(0) = 0$ and $\|\varphi_j\|_{C^1(B_j^2(0))} \leq L_j$. 

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2. \(d\) as the minimum distance between the small bodies \(\{D_1, D_2, \ldots, D_m\}\), i.e.
\[
d := \min_{1 \leq m, j \leq M} d_{mj},
\]
where \(d_{mj} := \text{dist}(D_m, D_j)\). We assume that
\[
0 < d \leq d_{\text{max}},
\]
and \(d_{\text{max}}\) is given.

3. \(\kappa_{\text{max}}\) as the upper bound of the used wave numbers, i.e. \(\kappa \in [0, \kappa_{\text{max}}]\).

The main theoretical result of this paper is the following theorem.

**Theorem 1.2.** There exist two positive constants \(a_0\) and \(c_0\) depending only on the Lipschitz character of \(B_m, m = 1, \ldots, M, d_{\text{max}}\) and \(\kappa_{\text{max}}\) such that if
\[
a \leq a_0 \quad \text{and} \quad \sqrt{M - 1} \frac{a}{d} \leq c_0\tag{1.8}
\]
then the far-field pattern \(U^\infty(\hat{x}, \theta)\) has the following asymptotic expansion
\[
U^\infty(\hat{x}, \theta) = \sum_{m=1}^{M} e^{-i\kappa \hat{x}z_m} Q_m + O \left( M\kappa a^2 + M(M-1) \left( \frac{\kappa a^3}{d} + \frac{a^3}{d^2} \right) + M(M-1)^2 a \left( \frac{\kappa a^3}{d} + \frac{a^3}{d^2} \right) \right),
\]
uniformly in \(\hat{x}\) and \(\theta\) in \(S^2\). The constant appearing in the estimate \(O(\cdot)\) depends only on the Lipschitz character of the obstacles, \(a_0, c_0, d_{\text{max}}\) and \(\kappa_{\text{max}}\). The coefficients \(Q_m, m = 1, \ldots, M\), are the solutions of the following linear algebraic system
\[
Q_m + \sum_{j \neq m}^{1} C_m \Phi(z_m, z_j) Q_j = -C_m U^1(z_m, \theta),
\]
for \(m = 1, \ldots, M\), with \(\Phi(z_m, z_j) := \frac{e^{-i\kappa|z_m-z_j|}}{4\pi|z_m-z_j|}, C_m := \int_{\partial D_m} \sigma_m(s) ds\) and \(\sigma_m\) is the solution of the integral equation of the first kind
\[
\int_{\partial D_m} \frac{\sigma_m(s)}{4\pi|t-s|} ds = 1, \quad t \in \partial D_m.
\]
The algebraic system (1.10) is invertible under the conditions:
\[
a \leq c_1 \quad \text{and} \quad \min_{j \neq m} \cos(\kappa|z_j - z_m|) \geq 0,\tag{1.12}
\]
where \(c_1\) depends only on Lipschitz character of the obstacles \(B_j, j = 1, \ldots, M\).

Before discussing this result compared to the existing literature, let us first mention the following remark.

**Remark 1.3.** The second condition of (1.8) can be replaced by the stronger one:
\[
(M - 1) \frac{a}{d} \leq c_2,
\]
with \(c_2\) depending only on the Lipschitz character of \(B_m, m = 1, \ldots, M\) and \(\kappa_{\text{max}}\), under which (1.9) is reduced to:
\[
U^\infty(\hat{x}, \theta) = \sum_{m=1}^{M} e^{-i\kappa \hat{x}z_m} Q_m + O \left( M\kappa a^2 + M(M-1) \left( \frac{\kappa a^3}{d} + \frac{a^3}{d^2} \right) \right),
\]
\[\text{if } \Omega \text{ is a domain containing the small bodies, and } \text{diam}(\Omega) \text{ denotes its diameter, then one example for the validity of the second condition in (1.12) is } \text{diam}(\Omega) < \frac{\sqrt{M}}{2\pi}.\]
and the algebraic system (1.10) is invertible as well. Remark that in this case the condition (1.7) is not required.

Due to the condition (1.13), the approximation (1.14) can also be reduced to

\[ U^\infty(\hat{x}, \theta) = \sum_{m=1}^{M} e^{-i\kappa z_m} Q_m + O\left( M \left( \kappa + \frac{1}{d} \right) a^2 \right). \]  

(1.15)

However this models only the Born approximation while the approximation in (1.14) takes into account some multiple scattering as the first order interaction, see Section 3 for more details.

This type of result, precisely the estimate (1.15), is known by A. Ramm since mid 1980’s, see [20–22] and the references therein for his recent related results. However, he used the (rough) condition \( \frac{a}{\kappa} \ll 1 \) and no mention has been made on the number of obstacles \( M \). In his arguments, he used the single layer potential representation (SLPR) of the scattered field. Using the same representation, we realize that a condition of the type \( (M - 1) \frac{a}{\kappa} \leq c \) is needed, see Proposition 2.2. However, using (naturally) the double layer potential representation (DLPR) we need only the weaker condition \( \sqrt{M - 1} \frac{a}{\kappa} \leq c_0 \), see Section 2.2. This condition appears naturally since we use the Neumann series expansion to estimate the inverse of the boundary operator, see the proof of Proposition 2.14.

The particular but important case where the obstacles have circular shapes has been considered recently by M. Cassier and C. Hazard in [11] where error estimates are obtained replacing the condition \( \sqrt{M - 1} \frac{a}{\kappa} \leq c_0 \) by the weaker one of the form \( \frac{a}{\kappa} \ll c_0 \) (appearing implicitly in their analysis). This is possible due to the use of the Fourier series expansion of the scattered fields with which they could avoid the use of the Neumann series expansion to estimate the inverse of the corresponding boundary operator. However, as it is mentioned in [11], they did not provide quantitative estimate of the errors in terms of the density of the obstacles (i.e. \( M \) and \( d \)).

Let us also mention the approach by V. Maz’ya, A. Movchan and M. Nieves [16, 17] where variational methods are used to study boundary value problems for the Laplacian with source terms in bounded domains. They obtain estimates in forms similar to the previous theorem with weaker conditions of the form \( \frac{a}{\kappa} \leq c \), or \( \frac{a}{\kappa} \leq c_0 \), where, here and in [11], \( d \) is the smallest distance between the centers of the scatterers). In their analysis, they rely on the maximum principle to treat the boundary estimates in addition to the fact that the source terms are assumed to be supported away from the small obstacles, a condition that can not be satisfied for scattering by incident plane waves. To avoid the use of the maximum principle, which is not valid due to the presence of the wave number \( \kappa \), we use boundary integral equation methods. The price to pay is the need of the stronger assumption \( \sqrt{M - 1} \frac{a}{\kappa} \leq c_0 \).

Let us finally mention that the integral equation methods are widely used in such a context, see for instance the series of works by H. Ammari and H. Kang and their collaborators, as [4] and the references therein. They combine layer potential techniques with the series expansion of the Green’s functions of the background medium to derive the full asymptotic expansion in terms of the polarization tensors. The difference between their asymptotic expansion and the one described in the previous theorem is that their polarization tensors are build up from densities which are solutions of a system of integral equations while in the previous theorem the approximating terms are build up from the linear algebraic system (1.10).

We notice that this algebraic system can be seen as a discrete approximation of the mentioned system of integral equations. Due to motivations coming from inverse problems, apart from few works as [3], they consider well separated scatterers and hence their asymptotic expansion are given only in terms of the size of the scatterers.

Before concluding the introduction, we find it is worth mentioning the following remark.

\textbf{Remark 1.4.} If we assume, in addition to the conditions of Theorem 1.2, that \( D_m \) are balls with same diameter \( a \) for \( m = 1, \ldots, M \), then we have the following asymptotic expansion of the far-field pattern:

\[ U^\infty(\hat{x}, \theta) = \sum_{m=1}^{M} e^{-i\kappa z_m} Q_m + \frac{a^3}{d^{\alpha-3\alpha}} + \frac{a^4}{d^{\alpha-4\alpha}} + M(M - 1) \left( \frac{a^3}{d^{2\alpha}} + \frac{a^4}{d^{4-\alpha}} + \frac{a^4}{d^{4-2\alpha}} \right) + M(M - 1) \frac{a^4}{d^{\alpha}} \]  

(1.16)
where $0 < \alpha \leq 1$.

Consider now the special case $d = a^t, M = a^{-s}$ with $t, s > 0$. Then the asymptotic expansion (1.16) can be rewritten as

$$U^\infty(\mathbf{x}, \theta) = \sum_{m=1}^{M} e^{-i\mathbf{k} \cdot \mathbf{z}_m} Q_m + O \left(a^{2-s} + a^{3-s-5t+3\alpha} + a^{4-s-9t+6\alpha} + a^{3-2s-2t\alpha} + a^{4-3s-3\alpha} + a^{4-2s-5t+2\alpha}\right).$$

As the diameter $a$ tends to zero the error term tends to zero for $t$ and $s$ such that $0 < t < 1$ and $0 < s < \min\{2(1-t), \frac{7-2a}{4}, \frac{12-9a}{12}, \frac{20-15a}{12}, \frac{4}{3} - t\alpha\}$. In particular for $t = \frac{1}{5}, s = 1$, we have

$$U^\infty(\mathbf{x}, \theta) = \sum_{m=1}^{M} e^{-i\mathbf{k} \cdot \mathbf{z}_m} Q_m + O \left(a + a^{2\alpha} + a^{3-\alpha} + a^{1+2\alpha}\right) = \sum_{m=1}^{M} e^{-i\mathbf{k} \cdot \mathbf{z}_m} Q_m + O \left(a^{\frac{1}{2}}\right) \quad [\text{obtained for } \alpha = \frac{1}{4}].$$

(1.17)

This particular case is used to derive the effective medium by perforation using many small bodies, see [21,22] for instance. The result of Remark 1.4, in particular (1.17), ensures the rate of the error in deriving such an effective medium.

Actually, the results (1.16) and (1.17) are valid for the non-flat Lipschitz domains $D_m = \epsilon B_m + z_m, m = 1, \ldots, M$ with the same diameter, i.e. $D_m$’s are Lipschitz domains and there exist constants $t_m \in (0, 1]$ such that

$$B_{t_m}^2(z_m) \subset D_m \subset B_{1}^2(z_m),$$

(1.18)

where $t_m$ are assumed to be uniformly bounded from below by a positive constant and $B_{r}^2(z)$ denotes the ball of center $z$ and radius $r$ in $\mathbb{R}^3$.

The rest of the paper is organized as follows. In section 2, we prove Theorem 1.2 in two steps. In section 2.1, we use single layer potential representation of the scattered field and show how the stronger condition $(M - 1) \frac{T}{2} \leq \epsilon$ appears naturally. In section 2.2, we use double layer potentials and reduce this condition to the weaker one $\sqrt{M-1} \frac{T}{2} \leq \epsilon_0$. In section 3, we deal with the inverse scattering problem which consists of recovering the location and the size of the small scatterers from farfield patterns corresponding to finitely many incident plane waves. We present numerical tests using the MUSIC algorithm with an emphasis on the multiple scattering effect due to closely spaced scatterers.

2. Proof of Theorem 1.2. We wish to warn the reader that in our analysis we use sometimes the parameter $\epsilon$ and some other times the parameter $a$ as they appear naturally in the estimates. But we bear in mind the relation (1.5) between $a$ and $\epsilon$.

2.1. Using SLP representation.

2.1.1. The representation via SLP. We start with the following proposition on the solution of the problem (1.1-1.3) via the single layer potential representation.

Proposition 2.1. For $m = 1, 2, \ldots, M$, there exists $\sigma_m \in L^2(\partial D_m)$ such that the problem (1.1-1.3) has a solution of the form

$$U^t(x) = U^i(x) + \sum_{m=1}^{M} \int_{\partial D_m} \Phi(x, s)\sigma_m(s) ds, \quad x \in \mathbb{R}^3 \setminus \left(\bigcup_{m=1}^{M} \overset{\shortparallel}{D_m}\right),$$

(2.1)

$$\Phi(x, y) := \frac{e^{i|\mathbf{x} - \mathbf{y}|}}{4\pi|\mathbf{x} - \mathbf{y}|}, \quad \text{for all } x, y \in \mathbb{R}^3. \quad \text{This solution is unique.}$$

Proof of Proposition 2.1. We look for the solution of the problem (1.1-1.3) of the form (2.1), then from the Dirichlet boundary condition (1.2), we obtain

$$\sum_{m=1}^{M} \int_{\partial D_m} \Phi(s_j, s)\sigma_m(s) ds = -U^i(s_j), \quad \forall s_j \in \partial D_j, \quad j = 1, \ldots, M.$$  

(2.2)
One can write it in a compact form as \((L + K)\sigma = -U^I\) with \(L := (L_{mj})_{m,j=1}^M\) and \(K := (K_{mj})_{m,j=1}^M\), where
\[
L_{mj} = \begin{cases} S_{mj} & \text{if } m = j \\ 0 & \text{else} \end{cases}, \quad K_{mj} = \begin{cases} S_{mj} & \text{if } m \neq j \\ 0 & \text{else} \end{cases},
\]
\(U^I = (U^I, \ldots, U^I)^T\) and \(\sigma = (\sigma_1, \ldots, \sigma_M)^T\). Here, for the indices \(m\) and \(j\) fixed, \(S_{mj}\) is the integral operator acting as
\[
S_{mj}(\sigma_j)(t) := \int_{\partial D_j} \Phi(t, s)\sigma_j(s) ds, \quad t \in \partial D_m. \tag{2.4}
\]

Then the operator \(S_{mm} : H^{-s}(\partial D_m) \to H^{1-s}(\partial D_m)\) is isomorphic and hence Fredholm with zero index and for \(m \neq j\), \(S_{mj} : H^{-s}(\partial D_j) \to H^{1-s}(\partial D_m)\) is compact for \(0 \leq s \leq 1\), see [19, Theorem 4.1].\(^3\) In our case we consider \(s = 0\). So, \((L + K) : \prod_{m=1}^M L^2(\partial D_m) \to \prod_{m=1}^M H^1(\partial D_m)\) is Fredholm with zero index. We induce the product of spaces by the maximum of the norms of the spaces. To show that \((L + K)\) is invertible it is enough to show that it is injective, i.e. \((L + K)\sigma = 0\) implies \(\sigma = 0\).

We write
\[
\hat{U}(x) = \sum_{m=1}^M \int_{\partial D_m} \Phi(x, s)\sigma_m(s) ds, \quad \text{in } \mathbb{R}^3 \setminus \left( \bigcup_{m=1}^M D_m \right)
\]
and
\[
\tilde{U}(x) = \sum_{m=1}^M \int_{\partial D_m} \Phi(x, s)\sigma_m(s) ds, \quad \text{in } \bigcup_{m=1}^M D_m.
\]

Then \(\hat{U}\) satisfies \(\Delta \hat{U} + \kappa^2 \hat{U} = 0\) for \(x \in \mathbb{R}^3 \setminus \left( \bigcup_{m=1}^M D_m \right)\), with S.R.C and \(\hat{U}(x) = 0\) on \(\bigcup_{m=1}^M \partial D_m\).

Similarly, \(\tilde{U}\) satisfies \(\Delta \tilde{U} + \kappa^2 \tilde{U} = 0\) for \(x \in \bigcup_{m=1}^M D_m\) with \(\tilde{U}(x) = 0\) on \(\bigcup_{m=1}^M \partial D_m\). From the uniqueness of the interior problem, which is true under the condition \(a < \frac{1}{\kappa_{\text{max}}} \frac{3}{\sqrt{4\pi}} j_{1/2,1}\), and the exterior problem we deduce that \(\hat{U} = 0\) and \(\tilde{U} = 0\) in their respective domains. Hence \(\frac{\partial \hat{U}}{\partial n}(x) = 0\) and \(\frac{\partial \tilde{U}}{\partial n}(x) = 0\) for \(x \in \bigcup_{m=1}^M \partial D_m\).

By the Jump relations, we have
\[
\frac{\partial \hat{U}}{\partial n}(x) = 0 \implies (K^\ast \sigma_m)(x) - \frac{\sigma_m(x)}{2} + \sum_{j \neq m} \frac{\partial}{\partial n_x} S_{mj}(\sigma_j)(x) = 0, \tag{2.5}
\]
and
\[
\frac{\partial \tilde{U}}{\partial n}(x) = 0 \implies (K^\ast \sigma_m)(x) + \frac{\sigma_m(x)}{2} + \sum_{j \neq m} \frac{\partial}{\partial n_x} S_{mj}(\sigma_j)(x) = 0. \tag{2.6}
\]

for \(x \in \partial D_m\) and for \(m = 1, \ldots, M\). Here, \(K^\ast\) is the adjoint of the double layer operator \(K\),
\[
(K \sigma_m)(x) := \int_{\partial D_m} \frac{\partial}{\partial n_s} \Phi(x, s)\sigma_m(s) ds, \quad \text{for } m = 1, \ldots, M.
\]

Difference between (2.5) and (2.6) provides us, \(\sigma_m = 0\) for all \(m\).

We conclude then that \(L + K := S : \prod_{m=1}^M L^2(\partial D_m) \to \prod_{m=1}^M H^1(\partial D_m)\) is invertible.

\(^3\)This property is proved for the case \(\kappa = 0\). By a perturbation argument, we can obtain the same results for every \(\kappa\) such that \(\kappa^2\) is not a Dirichlet-eigenvalue of \(-\Delta\) in \(D_m\). This last condition is satisfied for every \(\kappa\) fixed, \(\kappa \leq \kappa_{\text{max}}\), if we take \(a < \frac{1}{\kappa_{\text{max}}} \frac{3}{\sqrt{4\pi}} j_{1/2,1}\). Here \(j_{1/2,1}\) is the 1st positive zero of the Bessel function \(J_{1/2}\).
2.1.2. An appropriate estimate of the densities $\sigma_m$, $m = 1, \ldots, M$. From the above theorem, we have the following representation of $\sigma$:

$$
\sigma = (L + K)^{-1}U^f = L^{-1}(I + L^{-1}K)^{-1}U^f = L^{-1} \sum_{l=0}^{\infty} (-L^{-1}K)^l U^f, \text{ if } ||L^{-1}K|| < 1. \tag{2.7}
$$

The operator $L$ is invertible since it is Fredholm of index zero and injective (by the assumption $a < \frac{1}{\max \sqrt{\frac{d}{\epsilon}} 1_{1/2,1}}$). This implies that

$$
||\sigma|| \leq \frac{||L^{-1}||}{1 - ||L^{-1}|| ||K||} ||U^f||. \tag{2.8}
$$

Here we use the following notations:

\begin{align*}
||K|| := ||K||_{L^2(\partial D_m) \rightarrow H^1(\partial D_m)} &\equiv \max_{m=1}^{M} \sum_{j=1}^{M} ||K_{mj}||_{L^2(\partial D_j) \rightarrow H^1(\partial D_m))} \\
&= \max_{m=1}^{M} \sum_{j=1}^{M} ||S_{mj}||_{L^2(\partial D_j) \rightarrow H^1(\partial D_m))}, \tag{2.9}
\\
||L^{-1}|| := ||L^{-1}||_{H^1(\partial D_m) \rightarrow L^2(\partial D_m)} &\equiv \max_{m=1}^{M} \sum_{j=1}^{M} ||(L^{-1})_{mj}||_{H^1(\partial D_m) \rightarrow L^2(\partial D_j))} \\
&= \max_{m=1}^{M} ||S_{jm}^{-1}||_{L^2(\partial D_j) \rightarrow H^1(\partial D_m))}, \tag{2.10}
\\
||\sigma|| := ||\sigma||_{H^1(\partial D_m) \rightarrow L^2(\partial D_m)} &\equiv \max_{m=1}^{M} ||\sigma_m||_{H^1(\partial D_m) \rightarrow L^2(\partial D_m)} \tag{2.11}
\end{align*}

and

\begin{align*}
||U^f|| := ||U^f||_{H^1(\partial D_m) \rightarrow L^2(\partial D_m)} &\equiv \max_{m=1}^{M} ||U^f||_{H^1(\partial D_m) \rightarrow L^2(\partial D_m)} \tag{2.12}
\end{align*}

In the following proposition, we provide conditions under which $||L^{-1}|| ||K|| < 1$ and then estimate $||\sigma||$ via (2.8).

**Proposition 2.2.** There exists a constant $c$ depending only on the Lipschitz character of $B_m, m = 1, \ldots, M$, $d_{\max}$ and $\kappa_{\max}$ such that if $(M-1) \epsilon < c a^2$, then the $L^2$-norm of densities $\sigma_m$ appearing in the solution (2.1) of the problem (1.1-1.3) are uniformly bounded by a positive constant.

**Proof of Proposition 2.2.**

For any functions $f, g$ defined on $\partial D_x$ and $\partial B$ respectively, we define

$$
\tilde{f}(\xi) := f(\epsilon \xi + z) \quad \text{and} \quad \tilde{g}(x) := g \left( \frac{x - z}{\epsilon} \right).
$$

Let $T_1$ and $T_2$ be an orthonormal basis for the tangent plane to $\partial D_x$ at $x$ and let $\partial / \partial T = \sum_{l=1}^{2} \partial / \partial T_l T_l$, denote the tangential derivative on $\partial D_x$. We recall that the space $H^1(\partial D_x)$ is defined as

$$
H^1(\partial D_x) := \{ \phi \in L^2(\partial D_x); \partial \phi / \partial T \in L^2(\partial D_x) \}. \tag{2.13}
$$

We start with the following lemma which is similar to Lemma 4.1 of [7].

**Lemma 2.3.** Suppose $0 < \epsilon \leq 1$ and $D_\epsilon := \epsilon B + z \subset \mathbb{R}^n$. Then for each $\phi \in H^1(\partial D_x)$ and $\psi \in L^2(\partial D_x)$, we have

$$
||\psi||_{L^2(\partial D_x)} = \epsilon^{\frac{n+1}{2}} ||\tilde{\psi}||_{L^2(\partial B)} \tag{2.14}
$$
and
\[
\epsilon^{-\alpha} \| \hat{\phi} \|_{H^1(\partial B)} \leq \| \phi \|_{H^1(\partial D_\epsilon)} \leq \epsilon^{-\alpha/2} \| \hat{\phi} \|_{H^1(\partial B)}. \tag{2.15}
\]

**Proof of Lemma 2.3.**

Consider, \( \psi \in L^2(\partial D_\epsilon) \). Then (2.14) is derived as follows
\[
\| \psi \|_{L^2(\partial D_\epsilon)}^2 = \int_{\partial D_\epsilon} |\psi(x)|^2 dx
\]
\[
= \epsilon^{n-1} \int_{\partial B} |\psi(\epsilon \xi + z)|^2 d\xi
\]
\[
= \epsilon^{n-1} \int_{\partial B} |\hat{\psi}(\xi)|^2 d\xi, \quad (\hat{\psi}(\xi) := \psi(\epsilon \xi + z))
\]
\[
= \epsilon^{n-1} \| \hat{\psi} \|_{L^2(\partial B)}^2.
\]

Now consider, \( \phi \in H^1(\partial D_\epsilon) \). Then
\[
\| \phi \|_{H^1(\partial D_\epsilon)}^2 = \int_{\partial D_\epsilon} (|\phi(x)|^2 + |\partial T_\epsilon \phi(x)|^2) dx
\]
\[
= \epsilon^{n-1} \int_{\partial B} \left( \phi(\epsilon \xi + z)^2 + \frac{1}{\epsilon^2} |\partial T_\epsilon \phi(\epsilon \xi + z)|^2 \right) d\xi
\]
\[
= \epsilon^{n-1} \int_{\partial B} |\hat{\phi}(\xi)|^2 d\xi + \epsilon^{n-3} \int_{\partial B} |\partial T_\epsilon \hat{\phi}(\xi)|^2 d\xi.
\]

Using the fact that \( \epsilon^{n-1} \leq \epsilon^{n-3} \), we obtain (2.15).

We divide the rest of the proof of Proposition 2.2 into two steps. In the first step, we assume we have a single obstacle and then in the second step we deal with the multiple obstacle case.

**2.1.2.1. The case of a single obstacle.** Let us consider a single obstacle \( D_\epsilon := \epsilon B + z \). Then define the operator \( S_{D_\epsilon} : L^2(\partial D_\epsilon) \to H^1(\partial D_\epsilon) \) by
\[
(S_{D_\epsilon} \psi)(x) := \int_{\partial D_\epsilon} \Phi(x,y)\psi(y) dy. \tag{2.16}
\]

Following the arguments in the proof of Proposition 2.1, the integral operator \( S_{D_\epsilon} : L^2(\partial D_\epsilon) \to H^1(\partial D_\epsilon) \) is invertible. If we consider the problem (1.1-1.3) in \( \mathbb{R}^3 \setminus D_\epsilon \), we obtain
\[
\sigma = S_{D_\epsilon}^{-1} U^i, \quad \text{where } S_{D_\epsilon} = L + K,
\]
and then
\[
\| \sigma \|_{L^2(\partial D_\epsilon)} \leq \| S_{D_\epsilon}^{-1} \|_{L^2(\partial D_\epsilon), L^2(\partial D_\epsilon)} \| U^i \|_{L^2(\partial D_\epsilon)}.
\tag{2.17}
\]

**Lemma 2.4.** Let \( \phi \in H^1(\partial D_\epsilon) \) and \( \psi \in L^2(\partial D_\epsilon) \). Then,
\[
S_{D_\epsilon} \psi = \epsilon (S_B \psi)^\vee, \tag{2.18}
\]
\[
S_{D_\epsilon}^{-1} \phi = \epsilon^{-1} (S_B^{-1} \phi)^\vee.
\tag{2.19}
\]

and
\[
\| S_{D_\epsilon}^{-1} \|_{L^2(\partial D_\epsilon), L^2(\partial D_\epsilon)} \leq \epsilon^{-1} \| S_B^{-1} \|_{L^2(\partial B), L^2(\partial B)}.
\tag{2.20}
\]
with \( S_B^\epsilon \hat{\psi}(\xi) := \int_{\partial B} \frac{e^{i\epsilon |\xi - \eta|}}{4\pi |\xi - \eta|} \hat{\psi}(\eta) d\eta \).

**Proof of Lemma 2.4.** The identities (2.18) and (2.19) are derived respectively as follows

\[
S_{D,}\psi(x) = \int_{\partial D,} \frac{e^{i\epsilon |x - y|}}{4\pi |x - y|} \psi(y) dy = \int_{\partial B} \frac{e^{i\epsilon |\xi - \eta|}}{4\pi \epsilon |\xi - \eta|} \psi(\eta + z) \epsilon^2 d\eta = \epsilon S_B^\epsilon \hat{\psi}(\xi)
\]

and

\[
S_{D,}(S_B^{\epsilon - 1} \hat{\phi})^\vee = (S_B^{\epsilon - 1} \hat{\phi})^\vee = \epsilon \hat{\phi}^\vee = \epsilon \phi.
\]

To derive the estimate (2.20), we proceed as follows

\[
\left\| S_{D,}^{-1} \right\|_{L(H^1(\partial D_1),L^2(\partial D_2))} := \sup_{\hat{\phi} \in H^1(\partial D_1)} \left\| S_{D,}^{-1} \hat{\phi} \right\|_{L^2(\partial D_2)}
\]

(2.14), (2.15) \( \epsilon \left\| \hat{\phi} \right\|_{H^1(\partial B)} \)

(2.19) \( \epsilon \left\| \hat{\phi} \right\|_{H^1(\partial B)} \)

\[
= \epsilon^{-1} \left\| S_B^{\epsilon - 1} \hat{\phi} \right\|_{L^2(\partial B)}
\]

(2.20) \( \epsilon \left\| \hat{\phi} \right\|_{H^1(\partial B)} \)

From the explicit form of \( S_B^\epsilon \), we can estimate the left hand side of (2.20) by \( \epsilon^{-1} C \) using Banach’s theorem of uniform boundedness. However this theorem provides only an existence of \( C \) with no information on its dependence on \( B \) and \( \kappa \). The next lemma provides such an estimate.

**Lemma 2.5.** The operator norm of the inverse of \( S_{D,} : L^2(\partial D_1) \rightarrow H^1(\partial D_2) \), defined by \( S_{D,}\psi(x) := \int_{\partial D_1} \Phi(x,z) \psi(z) dz \) in (2.16), is estimated by \( \epsilon^{-1} \), i.e.

\[
\left\| S_{D,}^{-1} \right\|_{L(H^1(\partial D_1),L^2(\partial D_2))} \leq C_6 \epsilon^{-1},
\]

(2.21)

with \( C_6 := \frac{2\pi}{2\pi - (1 + \kappa)\epsilon |\partial B|} \left\| S_B^{\epsilon - 1} \right\|_{L(H^1(\partial B),L^2(\partial B))} \). Here, \( S_B^\epsilon : L^2(\partial B) \rightarrow H^1(\partial B) \) is the single layer potential with the wave number zero.

Here we should mention that if \( \epsilon \leq \frac{\pi}{(1 + \kappa)\epsilon |\partial B|} \left\| S_B^{\epsilon - 1} \right\|_{L(H^1(\partial B),L^2(\partial B))} \), then \( C_6 \) is bounded by

\[
2 \left\| S_B^{\epsilon - 1} \right\|_{L(H^1(\partial B),L^2(\partial B))},
\]

which is a constant depending only on \( \partial B \) through its Lipschitz character, see Remark 2.23.

**Proof of Lemma 2.5.** To estimate the operator norm of \( S_{D,}^{-1} \) we decompose \( S_{D,} = S_{D,}^{\kappa} + S_{D,}^{\delta} \) into two parts \( S_{D,}^{\kappa} \) (independent of \( \kappa \)) and \( S_{D,}^{\delta} \) (dependent of \( \kappa \)) given by

\[
S_{D,}^{\kappa}\psi(x) := \int_{\partial D_1} \frac{1}{4\pi |x - y|} \psi(y) dy,
\]

(2.22)

\[
S_{D,}^{\delta}\psi(x) := \int_{\partial D_1} \frac{e^{i\epsilon |x - y|} - 1}{4\pi |x - y|} \psi(y) dy.
\]

(2.23)

With this definition, \( S_{D,}^{\kappa} : L^2(\partial D_1) \rightarrow H^1(\partial D_2) \) is invertible, see [18,19]. Hence, \( S_{D,} = S_{D,}^{\kappa} \left( I + S_{D,}^{\kappa - 1} S_{D,}^{\delta} \right) \)

and so

\[
\left\| S_{D,}^{-1} \right\|_{L(H^1(\partial D_1),L^2(\partial D_2))} \leq \left\| \left( I + S_{D,}^{\kappa - 1} S_{D,}^{\delta} \right)^{-1} \right\|_{L(L^2(\partial D_1),L^2(\partial D_2))} \left\| S_{D,}^{\kappa - 1} \right\|_{L(H^1(\partial D_1),L^2(\partial D_2))}.
\]

(2.24)
So, to estimate the operator norm of $\mathcal{S}_{D_x}^{-1}$ one need to estimate the operator norm of $\left( I + S_{D_x}^{-1} S_{D_x}^{-1} \right)^{-1}$, in particular one need to have the knowledge about the operator norms of $S_{D_x}^{-1}$ and $S_{D_x}^{-1}$ to apply the Neumann series. For that purpose, we can estimate the operator norm of $S_{D_x}^{-1}$ from (2.20) by

$$
\left\| S_{D_x}^{-1} \right\|_{L(H^1(\partial D_x), L^2(\partial D_x))} \leq \epsilon^{-1} \left\| S_{D_x}^{-1} \right\|_{L(H^1(\partial D_x), L^2(\partial D_x))},
$$

(2.25)

Here $S_{D_x}^{-1}$ is defined as $S_{D_x}^{-1}(\xi) := \int_{\partial B} \frac{1}{4\pi |\xi - \eta|} \hat{\psi}(\eta) d\eta$. From the definition of the operator $S_{D_x}$ in (2.23), we deduce that

$$
S_{D_x}^{-1}(x) = \epsilon \int_{\partial B} e^{i\kappa|\xi - \eta|} - 1 \hat{\psi}(\eta) d\eta
$$

and so the tangential derivative w.r.t. basic vector $T$ ($T_1$ or $T_2$) is given by

$$
\frac{\partial}{\partial T} S_{D_x}^{-1}(x) = \int_{\partial B} \left( \frac{e^{i\kappa|x-y|}}{4\pi |x-y|} - \frac{1}{|x-y|} \right) \frac{x-y}{4\pi} \cdot T \hat{\psi}(\eta) d\eta
$$

$$
= \int_{\partial B} \left( \frac{e^{i\kappa|\xi-\eta|}}{4\pi e^{1/\eta}} - \frac{1}{e^{1/\eta}} \right) \frac{\xi-\eta}{4\pi e^{1/\eta}} \cdot T \hat{\psi}(\eta) d\eta
$$

$$
= \int_{\partial B} \frac{(i\kappa)^2}{4\pi} \sum_{l=1}^{\infty} \left( \frac{1}{l} - \frac{1}{l+1} \right) (i\kappa)^{l-1} |\eta|^{l-1} \frac{\xi-\eta}{|\eta|} \cdot T \hat{\psi}(\eta) d\eta
$$

(2.27)

where we used the following expansions

$$
e^{i\kappa|\xi-\eta|} = \frac{i\kappa}{4\pi |\xi-\eta|} + \frac{1}{2!} \frac{(i\kappa)^2}{4\pi} |\xi-\eta| + \ldots
$$

$$
= \frac{i\kappa}{4\pi |\xi-\eta|} + \frac{(i\kappa)^2}{4\pi} \sum_{l=1}^{\infty} \frac{(i\kappa)^{l-1}}{l!} |\eta|^{l-1}
$$

and

$$
e^{i\kappa|\xi-\eta|} - 1 = \frac{1}{4!} \frac{i\kappa^2}{4\pi |\xi-\eta|^2} + \frac{1}{2!} \frac{(i\kappa)^2}{4\pi} |\xi-\eta| + \ldots
$$

$$
= \frac{i\kappa}{4\pi |\xi-\eta|} + \frac{(i\kappa)^2}{4\pi} \sum_{l=1}^{\infty} \frac{(i\kappa)^{l-1}}{l+1!} |\eta|^{l-1}.\]

From (2.26), we observe that

$$
|S_{D_x}^{-1}(\xi)| \leq \epsilon \left\| \frac{e^{i\kappa|\xi-\eta|} - 1}{4\pi |\xi-\eta|} \hat{\psi} \right\|_{L^2(\eta)}
$$

$$
= \frac{\epsilon}{4\pi} \left\| \sum_{l=1}^{\infty} \frac{(i\kappa)^{l-1}}{l!} |\xi-\eta|^{l-1} \hat{\psi} \right\|_{L^2(\eta)}
$$

$$
\leq \frac{\kappa^2}{4\pi} \sum_{l=1}^{\infty} \frac{(i\kappa)^{l-1}}{l!} |\xi-\eta|^{l-1} \left\| \hat{\psi} \right\|_{L^2(\eta)}
$$

(Since, $\left\| x \right\|_{L^2(D)} \leq \left\| x \right\|_{L^2(\partial B)} |D|^{1/2}$)
\[
\leq \frac{\kappa^2}{4\pi} |\partial B|^{\frac{1}{2}} \sum_{i=1}^{\infty} \left[ \frac{1}{2} \kappa \epsilon \|\xi - \cdot\|_{L^2(\partial B)} |\partial B|^\frac{3}{2} \right]^{l-1} \|\hat{\psi}\|^2_{L^2(\partial B)} \\
= \frac{\kappa^2}{4\pi} |\partial B|^{\frac{1}{2}} \frac{\|\hat{\psi}\|^2_{L^2(\partial B)}}{1 - \frac{1}{2} \kappa \epsilon \|\xi - \cdot\|_{L^2(\partial B)} |\partial B|^\frac{3}{2}}, \text{ if } \left( \epsilon < \frac{2}{\kappa_{\text{max}} \max_m \text{diam}(B_m)} \right) \equiv (\kappa_{\text{max}} a < 2) \\
\leq \frac{\kappa^2}{2\pi} |\partial B|^{\frac{1}{2}} \|\hat{\psi}\|^2_{L^2(\partial B)}, \text{ for } \left( \epsilon \leq \frac{1}{\kappa_{\text{max}} \max_m \text{diam}(B_m)} \right) \equiv (\kappa_{\text{max}} a \leq 1). \quad (2.28)
\]

We set \( C_1 := \frac{|\partial B|}{2\pi} \). From this we get,
\[
\|S_{D, e}^d \psi\|^2_{L^2(\partial D_e)} = \int_{\partial D_e} \|S_{D, e}^d \psi(x)\|^2 dx \\
\leq (2.28) \int_{\partial D_e} \left[ C_1 \kappa^2 \|\hat{\psi}\|^2_{L^2(\partial B)} \right]^2 dx \\
= C_1^2 \kappa^2 \epsilon^4 \|\hat{\psi}\|^2_{L^2(\partial B)} \int_{\partial D_e} dx \\
= C_1^2 \kappa^2 |\partial B| \epsilon^6 \|\hat{\psi}\|^2_{L^2(\partial B)},
\]
which gives us
\[
\|S_{D, e}^d \psi\|_{L^2(\partial D_e)} \leq C_1 \kappa^2 |\partial B|^{\frac{1}{2}} \|\hat{\psi}\|_{L^2(\partial B)}. \quad (2.29)
\]

From (2.27), we have
\[
\frac{\partial}{\partial T} S_{D, e}^d \psi(x) = -\frac{\kappa^2}{4\pi} \int_{\partial B} \left[ \sum_{l=1}^{\infty} \left( \frac{1}{l!} - \frac{1}{l + 1!} \right) (i\kappa)^{l-1} \|\hat{\psi}\|^2_{L^2(\partial B)} \right] \frac{\xi - \eta}{|\xi - \eta|} \cdot T \psi(\eta) d\eta,
\]
which gives us
\[
\left| \frac{\partial}{\partial T} S_{D, e}^d \psi(x) \right| \leq \frac{\kappa^2}{4\pi} \left| \sum_{l=1}^{\infty} \left( \frac{1}{l!} - \frac{1}{l + 1!} \right) (i\kappa)^{l-1} \|\hat{\psi}\|^2_{L^2(\partial B)} \right| \\
\leq \frac{\kappa^2}{4\pi} \left| \sum_{l=1}^{\infty} \frac{l}{l + 1} (i\kappa)^{l-1} \|\hat{\psi}\|^2_{L^2(\partial B)} \right| \\
\leq \frac{\kappa^2}{4\pi} \sum_{l=1}^{\infty} \frac{l}{l + 1} \frac{2}{\kappa^2} \left[ \frac{\|\hat{\psi}\|^2_{L^2(\partial B)}}{1 - \frac{1}{2} \kappa \epsilon \|\xi - \cdot\|_{L^2(\partial B)} |\partial B|^\frac{3}{2}} \right]^{l-1} \|\hat{\psi}\|^2_{L^2(\partial B)} \\
= \frac{\kappa^2}{8\pi} \|\hat{\psi}\|_{L^2(\partial B)} \frac{1}{1 - \frac{1}{2} \kappa \epsilon \|\xi - \cdot\|_{L^2(\partial B)} |\partial B|^\frac{3}{2}}, \text{ if } \left( \epsilon < \frac{2}{\kappa_{\text{max}} \max_m \text{diam}(B_m)} \right) \equiv (\kappa_{\text{max}} a < 2) \\
\leq \frac{\kappa^2}{4\pi} \|\hat{\psi}\|_{L^2(\partial B)} \|\hat{\psi}\|_{L^2(\partial B)} \|\hat{\psi}\|_{L^2(\partial B)} \left[ \frac{1}{\kappa_{\text{max}} \max_m \text{diam}(B_m)} \right] \equiv (\kappa_{\text{max}} a \leq 1), \\
= \frac{1}{2} C_1 \kappa^2 \epsilon^2 \|\hat{\psi}\|_{L^2(\partial B)}. \quad (2.30)
\]

From this we obtain,
\[
\left| \frac{\partial}{\partial T} S_{D, e}^d \psi \right|_{L^2(\partial D_e)} = \int_{\partial D_e} \left| \frac{\partial}{\partial T} S_{D, e}^d \psi(x) \right|^2 dx \\
\leq \frac{3}{2} \int_{\partial D_e} \left[ \frac{1}{2} C_1 \kappa^2 \epsilon^2 \|\hat{\psi}\|_{L^2(\partial B)} \right]^2 dx.
\]
By substituting the above and (2.25) in (2.24), we obtain the required result (2.21).

We estimate the norm of the operator $S_{D_i}^{de}$ as

$$
\left\| \frac{\partial}{\partial T} S_{D_i}^{de} \psi \right\|_{L^2(\partial D_i)} \leq \frac{1}{2} C_1 \kappa^2 \epsilon^3 |\partial B|^\frac{1}{2} \|\psi\|_{L^2(\partial B)}.
$$

Hence

$$
\left\| \frac{\partial}{\partial T} S_{D_i}^{de} \psi \right\|_{L^2(\partial D_i)} \leq \frac{1}{2} C_1 \kappa^2 \epsilon^3 |\partial B|^\frac{1}{2} \|\psi\|_{L^2(\partial B)}.
$$

Now, we have

$$
\left\| S_{D_i}^{de} \psi \right\|_{H^1(\partial D_i)} = \left\| S_{D_i}^{de} \psi \right\|_{L^2(\partial D_i)} + \sum_{i=1}^{2} \left\| \frac{\partial}{\partial T} S_{D_i}^{de} \psi \right\|_{L^2(\partial D_i)}
$$

and so from (2.29) and (2.31), we can write

$$
\left\| S_{D_i}^{de} \psi \right\|_{H^1(\partial D_i)} \leq C_1 \kappa^3 |\partial B|^\frac{1}{2} \|\psi\|_{L^2(\partial B)} + C_1 \kappa^3 |\partial B|^\frac{1}{2} \|\psi\|_{L^2(\partial B)} = C_1 (1 + \kappa) \kappa |\partial B|^\frac{1}{2} \|\psi\|_{L^2(\partial B)}.
$$

We estimate the norm of the operator $S_{D_i}^{de}$ as

$$
\left\| S_{D_i}^{de} \right\|_{L^2(\partial D_i), H^1(\partial D_i)} = \sup_{\psi(\not\equiv 0) \in L^2(\partial D_i)} \left\| S_{D_i}^{de} \psi \right\|_{H^1(\partial D_i)} \leq \sup_{\psi(\not\equiv 0) \in L^2(\partial B)} C_1 (1 + \kappa) \kappa |\partial B|^\frac{1}{2} \|\psi\|_{L^2(\partial B)}.
$$

Hence, we get

$$
\left\| S_{D_i}^{de} \right\|_{L^2(\partial D_i), L^2(\partial D_i)} \leq \left\| S_{D_i}^{de} \right\|_{L^2(\partial D_i), L^2(\partial D_i)} \left\| S_{D_i}^{de} \right\|_{L^2(\partial D_i), H^1(\partial D_i)} 
$$

$$
\leq C_2 (1 + \kappa) \kappa |\partial B|^\frac{1}{2},
$$

where $C_2 := C_1 |\partial B|^\frac{1}{2} \left\| S_{B}^{-1} \right\|_{L^2(H^1(\partial B), L^2(\partial B))} = \frac{|\partial B|}{2 \kappa} \left\| S_{B}^{-1} \right\|_{L^2(H^1(\partial B), L^2(\partial B))}$. Assuming $\epsilon$ to satisfy the condition $\epsilon < \frac{1}{C_2 (1 + \kappa) \kappa}$, then $\left\| S_{D_i}^{de} \right\|_{L^2(\partial D_i), L^2(\partial D_i)} < 1$ and hence by using the Neumann series we obtain the following

$$
\left\| (I + S_{D_i}^{de})^{-1} \right\|_{L^2(\partial D_i), L^2(\partial D_i)} \leq \frac{1}{1 - \left\| S_{D_i}^{de} \right\|_{L^2(\partial D_i), L^2(\partial D_i)}} \leq \frac{1}{C_3}.
$$

By substituting the above and (2.25) in (2.24), we obtain the required result (2.21).
2.1.2.2. The multiple obstacle case.

Lemma 2.6. For \( m, j = 1, 2, \ldots, M \), the operator \( S_{mj} : L^2(\partial D_j) \to H^1(\partial D_m) \) defined by

\[
S_{mj}(\sigma_j)(s_m) := \int_{\partial D_j} \Phi(s_m, s_m) \sigma_j(s) ds, \ s_m \in \partial D_m
\]

enjoys the following estimates,

- For \( j = m \),

\[
||S_{mm}^{-1}||_{L(H^1(\partial D_m), L^2(\partial D_m))} \leq C_{6m} \epsilon^{-1},
\]

where \( C_{6m} := \frac{2\pi ||S_{mm}^{*^{-1}}||_{L(H^1(\partial B_m), L^2(\partial B_m))}}{2\pi - (1+\kappa)\epsilon ||\partial B_m||} \).

- For \( j \neq m \),

\[
||S_{mj}||_{L(L^2(\partial D_j), H^1(\partial D_m))} \leq \frac{1}{4\pi} \left( \frac{2b_1 + 2}{d^2} \right) ||\partial B_j|\epsilon^2,
\]

where \( ||\partial B_j|| := \max_{m} ||\partial B_m|| \).

Proof of Lemma 2.6. The estimate (2.36) is nothing else but (2.21) of Lemma 2.5, replacing \( B \) by \( B_m \), \( z \) by \( z_m \) and \( D_m \) by \( D_m \) respectively. It remains to prove the estimate (2.37).

We have

\[
||S_{mj}(\psi)||_{L(L^2(\partial D_j), H^1(\partial D_m))} = \sup_{\psi(\neq 0) \in L^2(\partial D_j)} \frac{||S_{mj}(\psi)||_{H^1(\partial D_m)}}{||\psi||_{L^2(\partial D_j)}} = \sup_{\psi(\neq 0) \in L^2(\partial D_j)} \frac{||S_{mj}(\psi)||_{L^2(\partial D_m)} + ||\partial T \psi||_{L^2(\partial D_m)}}{||\psi||_{L^2(\partial D_j)}}.
\]

Let \( \psi \in L^2(\partial D_j) \) then for \( x \in \partial D_m \), we have

\[
|S_{mj}(\psi)(x)| = \left| \int_{\partial D_j} \Phi(x, s) \psi(s) ds \right| \\
\leq \int_{\partial D_j} \frac{1}{4\pi |x - s|} |\psi(s)| ds \\
\leq \frac{1}{4\pi d_{mj}} \int_{\partial D_j} |\psi(s)| ds (d_{mj} := \text{dist}(D_m, D_j)) \\
\leq \frac{1}{4\pi d_{mj}} \epsilon |\partial B_j|^{\frac{1}{2}} ||\psi||_{L^2(\partial D_j)},
\]

from which, we obtain

\[
||S_{mj}(\psi)||_{L^2(\partial D_m)} = \left( \int_{\partial D_m} |S_{mj}(\psi)(x)|^2 dx \right)^{\frac{1}{2}} \\
\leq \left( \int_{\partial D_m} \frac{1}{4\pi d_{mj}} \epsilon |\partial B_j|^{\frac{1}{2}} ||\psi||_{L^2(\partial D_j)} \left( \int_{\partial D_m} dx \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \\
= \frac{1}{4\pi d_{mj}} \epsilon^2 |\partial B_j|^{\frac{1}{2}} ||\partial B_m|^{\frac{1}{2}} ||\psi||_{L^2(\partial D_j)}.
\]

Now for \( T = T_1 \) and \( T_2 \), we have

\[
\left| \frac{\partial}{\partial T} S_{mj}(\psi)(x) \right| = \left| \int_{\partial D_j} \frac{\partial}{\partial T} \Phi(x, s) \psi(s) ds \right|
\]
\[ = \int_{\partial D_j} \nabla_x \Phi(x, s) \cdot T_x \psi(s) \, ds \]
\[ \leq \int_{\partial D_j} |\nabla_x \Phi(x, s)| \cdot |\psi(s)| \, ds \]
\[ = \int_{\partial D_j} \left| \frac{e^{i\kappa |x-s|}}{4\pi |x-s|} \right| \frac{x-s}{|x-s|} \cdot |\psi(s)| \, ds \]
\[ \leq \int_{\partial D_j} \frac{1}{4\pi |x-s|} \left( \kappa + \frac{1}{|x-s|} \right) |\psi(s)| \, ds \]
\[ \leq \frac{1}{4\pi} \left( \frac{\kappa}{d_{mj}} + \frac{1}{d_{mj}^2} \right) \int_{\partial D_j} |\psi(s)| \, ds \]
\[ \leq \frac{1}{4\pi} \left( \frac{\kappa}{d_{mj}} + \frac{1}{d_{mj}^2} \right) \epsilon |\partial B_j| \frac{1}{2} ||\psi||_{L^2(\partial D_j)}. \] (2.41)

From which, we obtain

\[ \left\| \frac{\partial}{\partial T} S_{mj} \psi \right\|_{L^2(\partial D_m)} = \left( \int_{\partial D_m} \left( \frac{\partial}{\partial T} S_{mj} \psi(x) \right)^2 \, dx \right)^{\frac{1}{2}} \]
\[ \leq \frac{1}{4\pi} \left( \frac{\kappa}{d_{mj}} + \frac{1}{d_{mj}^2} \right) \epsilon |\partial B_j| \frac{1}{2} \left( \int_{\partial D_m} \, dx \right)^{\frac{1}{2}} \]
\[ = \frac{1}{4\pi} \left( \frac{\kappa}{d_{mj}} + \frac{1}{d_{mj}^2} \right) \epsilon^2 |\partial B_j| \frac{1}{2} ||\psi||_{L^2(\partial D_j)}. \] (2.42)

From (2.13), (2.40) and (2.42), we derive

\[ ||S_{mj}\psi||_{H^1(\partial D_m)} \leq \frac{1}{4\pi} \left( \frac{2\kappa + 1}{d_{mj}} + \frac{2}{d_{mj}^2} \right) \epsilon^2 \left| \partial B_j \right| \frac{1}{2} ||\psi||_{L^2(\partial D_j)}. \] (2.43)

Substitution of (2.43) in (2.38) gives us

\[ ||S_{mj}||_{L^2(\partial D_j), H^1(\partial D_m)} \leq \frac{1}{4\pi} \left( \frac{2\kappa + 1}{d_{mj}} + \frac{2}{d_{mj}^2} \right) \epsilon^2 \left| \partial B_j \right| \frac{1}{2} \left| \partial B_m \right| \frac{1}{2} \leq \frac{1}{4\pi} \left( \frac{2\kappa + 1}{d} + \frac{2}{d^2} \right) |B| \epsilon^2. \]

**End of the proof of Proposition 2.2.**

By substituting (2.36) in (2.10) and (2.37) in (2.9), we obtain

\[ ||K|| \equiv \max_{m=1}^{M} \sum_{j=1}^{M} ||S_{mj}||_{L^2(\partial D_j), H^1(\partial D_m)} \]
\[ \leq \frac{M - 1}{4\pi} \left( \frac{2\kappa + 1}{d} + \frac{2}{d^2} \right) |B| \epsilon^2 \] (2.44)

and

\[ ||L^{-1}|| \equiv \max_{m=1}^{M} \left| S_{mn}^{-1} \right|_{L^2(\partial D_m), H^1(\partial D_m)} \]
\[ \equiv \max_{m=1}^{M} C_0 \epsilon^{-1}. \] (2.45)
Hence, (2.44) and (2.45) jointly provide
\[
\|L^{-1}\| \|K\| \leq \frac{M - 1}{4\pi} \left( \max_{m=1}^{M} C_{6m} \right) \|\partial B\| \left( \frac{2\kappa + 1}{d} + \frac{2}{d^2} \right) \epsilon, \tag{2.46}
\]
By imposing the condition \(\|L^{-1}\| \|K\| < 1\), we have from (2.8) and (2.11-2.12);
\[
\|\sigma_{m}\|_{L^2(\partial D_m)} \leq \|\sigma\| \leq \frac{\|L^{-1}\|}{1 - \|L^{-1}\| \|K\|} \|U^I\|
\leq C_p \|L^{-1}\| \max_{m=1}^{M} \|U^I\|_{H^1(\partial D_m)} \left( \frac{1}{1 - C_s \epsilon}, \text{ a positive constant} \right)
\]
for all \(m \in \{1, 2, \ldots, M\}\). But,
\[
\|U^I\|_{H^1(\partial D_m)} = \|U^I\|_{L^2(\partial D_m)} + \|\partial T U^I\|_{L^2(\partial D_m)}
\leq \epsilon |\partial B_m|^\frac{1}{2} + 2\epsilon |\partial B_m|^\frac{1}{2}
\leq (1 + 2\epsilon) |\partial B_m|^\frac{1}{2}, \forall m = 1, 2, \ldots, M. \tag{2.48}
\]
Now by substituting (2.48) in (2.47), for each \(m = 1, 1, \ldots, M\), we obtain
\[
\|\sigma_{m}\|_{L^2(\partial D_m)} \leq C(\kappa), \tag{2.49}
\]
where \(C(\kappa) := C |\partial B|^{\frac{1}{2}} (1 + 2\kappa)\).
The condition \(\|L^{-1}\| \|K\| < 1\) is satisfied if
\[
C_s = \frac{M - 1}{4\pi} |\partial B| \left( \frac{2\kappa + 1}{d} + \frac{2}{d^2} \right) \left( \max_{m=1}^{M} C_{6m} \right) \epsilon < 1. \tag{2.50}
\]
Since \((2\kappa + 1)d < \bar{c}\) for \(\bar{c} := (2\kappa_{\text{max}} + 1)d_{\text{max}}\), then (2.50) reads as \((M - 1)\epsilon < c d^2\), where we set
\[
\epsilon := \left[ \frac{(\bar{c} + 2)}{4\pi} |\partial B| \max_{m=1}^{M} C_{6m} \right]^{-1}.
\]

2.1.3. Further estimates on the total charge \(\int_{\partial D_m} \sigma_{m}(s) ds, m = 1, \ldots, M\).

**Definition 2.7.** Following [20], we call \(\sigma_{m} \in L^2(\partial D_m)\) used in (2.1), the solution of the problem (1.1-1.3), the surface charge distributions. For these surface charge distributions, we define the total charge on each surface \(\partial D_m\) denoted by \(Q_m\) as
\[
Q_m := \int_{\partial D_m} \sigma_{m}(s) ds. \tag{2.51}
\]

**Lemma 2.8.** For \(m = 1, 2, \ldots, M\), we have
\[
|Q_m| \leq \bar{c} \epsilon, \tag{2.52}
\]
where \(\bar{c} := |\partial B|C(1 + 2\kappa)\) with \(\partial B\) and \(C\) are defined in (2.37) and (2.47) respectively.

**Proof of Lemma 2.8.** From Proposition 2.2, we know that the surface charge distributions \(\sigma_{m} \in L^2(\partial D_m)\) are bounded by a constant \(C(\kappa) := C |\partial B|^{\frac{1}{2}} (1 + 2\kappa)\). Hence
\[
|Q_m| = \left| \int_{\partial D_m} \sigma_{m}(s) ds \right|
\]
On the Foldy-Lax approximation of the scattering by small bodies

\[ \leq \|1\|_{L^2(\partial D_m)} \|\sigma_m\|_{L^2(\partial D_m)} \]
\[ \leq \|1\|_{L^2(\partial D_m)} C |\partial B|^\frac{1}{2} (1 + 2\kappa) \]
\[ \leq |\partial B| C(1 + 2\kappa) \epsilon. \]

**Proposition 2.9.** The far-field pattern \( U^\infty \) of the scattered solution of the problem (1.1-1.3) has the following asymptotic expansion

\[ U^\infty(\hat{x}) = \sum_{m=1}^{M} [e^{-i\kappa \cdot \hat{x} - z_m} Q_m + O(\kappa a^2)], \tag{2.53} \]

with \( Q_m \) given by (2.51), if \( \kappa a < 1 \) where \( O(\kappa a^2) \leq C\kappa a^2 \) and \( C := \frac{|\partial B| C(1 + 2\kappa)}{\max_{1 \leq m \leq M} \text{diam}(B_m)}. \)

**Proof of Proposition 2.9.** From (2.1), we have

\[ U^s(x) = \sum_{m=1}^{M} \int_{\partial D_m} \Phi(x, s) \sigma_m(s) ds, \quad \text{for} \quad x \in \mathbb{R}^3 \setminus \left( \bigcup_{m=1}^{M} D_m \right). \]

Hence

\[ U^\infty(\hat{x}) = \sum_{m=1}^{M} \int_{\partial D_m} e^{-i\kappa \cdot \hat{x} - z_m} \sigma_m(s) ds \]
\[ = \sum_{m=1}^{M} \left( \int_{\partial D_m} e^{-i\kappa \cdot \hat{x} - z_m} \sigma_m(s) ds + \int_{\partial D_m} [e^{-i\kappa \cdot \hat{x} - z_m} - e^{-i\kappa \cdot \hat{x} - z_m}] \sigma_m(s) ds \right) \]
\[ = \sum_{m=1}^{M} \left( e^{-i\kappa \cdot \hat{x} - z_m} Q_m + \int_{\partial D_m} [e^{-i\kappa \cdot \hat{x} - z_m} - e^{-i\kappa \cdot \hat{x} - z_m}] \sigma_m(s) ds \right). \tag{2.54} \]

As in Lemma 2.8, we have from Proposition 2.2;

\[ \int_{\partial D_m} |\sigma_m(s)| ds \leq C a, \tag{2.55} \]

with \( C := \frac{\tilde{c}}{\max_{1 \leq m \leq M} \text{diam}(B_m)} = \frac{|\partial B| C(1 + 2\kappa)}{\max_{1 \leq m \leq M} \text{diam}(B_m)} \). \tag{2.56} \]

It gives us the following estimate;

\[ \left| \int_{\partial D_m} [e^{-i\kappa \cdot \hat{x} - z_m} - e^{-i\kappa \cdot \hat{x} - z_m}] \sigma_m(s) ds \right| \leq \int_{\partial D_m} \left| e^{-i\kappa \cdot \hat{x} - z_m} - e^{-i\kappa \cdot \hat{x} - z_m} \right| |\sigma_m(s)| ds \]
\[ \leq \int_{\partial D_m} \sum_{l=1}^{\infty} \kappa^l |s - z_m| |\sigma_m(s)| ds \]
\[ \leq \int_{\partial D_m} \sum_{l=1}^{\infty} \kappa^l \left( \frac{\tilde{a}}{2} \right)^l |\sigma_m(s)| ds \]
\[ \leq (2.55) C a \sum_{l=1}^{\infty} \kappa^l \left( \frac{\tilde{a}}{2} \right)^l \]
\[ = \frac{1}{2} C \kappa a^2 \frac{1}{1 - \kappa^2} \quad \text{if} \quad a < \frac{2}{\kappa_{\max}} \left( \leq \frac{2}{\kappa} \right) \tag{2.57} \]
which means
\[
\int_{\partial D_m} [e^{-i\kappa \cdot s} - e^{-i\kappa \cdot z_m}]\sigma_m(s)ds \leq C\kappa a^2, \text{ for } a \leq \frac{1}{\kappa_{\text{max}}}. \tag{2.58}
\]

Now substituition of (2.58) in (2.54) gives the required result (2.53).

Let us derive a formula for $Q_m$. For $s_m \in \partial D_m$, using the Dirichlet boundary condition (1.2), we have
\[
0 = U^i(s_m) = U^i(s_m) + \sum_{j=1}^{M} \int_{\partial D_j} \Phi(s_m, s)\sigma_j(s)ds
\]
\[
= U^i(s_m) + \sum_{j=1}^{M} \left( \Phi(s_m, z_j)Q_j + \int_{\partial D_j} |\Phi(s_m, s) - \Phi(s_m, z_j)|\sigma_j(s)ds \right) + \int_{\partial D_m} \Phi(s_m, s)\sigma_m(s)ds.
\tag{2.59}
\]

To estimate $\int_{\partial D_j} |\Phi(s_m, s) - \Phi(s_m, z_j)|\sigma_j(s)ds$ for $j \neq m$, we write from Taylor series that
\[
\Phi(s_m, s) - \Phi(s_m, z_j) = (s - z_j) \cdot R(s_m, s), \quad R(s_m, s) = \int_0^1 \nabla_2 \Phi(s_m, s - \alpha(s - z_j))d\alpha.
\tag{2.60}
\]

- From the definition of $\Phi(x, y) := \frac{e^{i|\kappa| - x - y}}{4\pi|x-y|}$, we have $\nabla_2 \Phi(x, y) = \Phi(x, y) \left[ \frac{1}{|x-y|} - i\kappa \right] \frac{x-y}{|x-y|}$ and hence, for $s \in D_j$, we obtain
\[
|R(s_m, s)\| \leq \max_{y \in \overline{D_j}} |\nabla_2 \Phi(s_m, y)| < \frac{1}{d} \left( \kappa + \frac{1}{d} \right).
\tag{2.61}
\]

For $m, j = 1, \ldots, M$, and $j \neq m$, by making use of (2.61) and (2.55) we obtain the estimate below;
\[
\left| \int_{\partial D_j} |\Phi(s_m, s) - \Phi(s_m, z_j)|\sigma_j(s)ds \right| = \left| \int_{\partial D_j} (s - z_j) \cdot R(s_m, s)\sigma_j(s)ds \right|
\leq \int_{\partial D_j} |s - z_j| |R(s_m, s)| |\sigma_j(s)| ds
\leq \frac{a}{d} \left( \kappa + \frac{1}{d} \right) \int_{\partial D_j} |\sigma_j(s)| ds
< C\frac{a}{d} \left( \kappa + \frac{1}{d} \right) a.
\tag{2.62}
\]

Define, $\Phi_0(s_m, s) := \frac{1}{4\pi|s_m - s|}$. Then (2.59) can be written as
\[
0 = U^i(s_m) + \sum_{j=1}^{M} \Phi(s_m, z_j)Q_j + O \left( (M-1) \left( \frac{\kappa a^2}{d} + \frac{a^2}{d^2} \right) \right)
\tag{2.63}
\]
\[
+ \int_{\partial D_m} \Phi_0(s_m, s) \left[ 1 + (e^{i|\kappa| |s_m - s|} - 1) \right] \sigma_m(s)ds.
\]

By using the Taylor series expansions of the exponential term $e^{i|\kappa| |s_m - s|}$, the above can also be written as,
\[
\int_{\partial D_m} \Phi_0(s_m, s)\sigma_m(s)ds + O(\kappa a) = -U^i(s_m) - \sum_{j=1}^{M} \Phi(s_m, z_j)Q_j + O \left( (M-1) \left( \frac{\kappa a^2}{d} + \frac{a^2}{d^2} \right) \right). \tag{2.64}
\]

Indeed,
For $m = 1, \ldots, M$, we have
\[
\left| \int_{\partial D_m} \Phi_0(s_m, s) \left( e^{i\kappa |s_m - s|} - 1 \right) \sigma_m(s) ds \right| = \left| \int_{\partial D_m} \frac{1}{4\pi |s_m - s|} \left( \sum_{l=1}^{\infty} \frac{(ik)^l}{l!} |s_m - s|^l \right) \sigma_m(s) ds \right|
\begin{align*}
&= \frac{1}{4\pi} \int_{\partial D_m} \left( \sum_{l=1}^{\infty} \frac{(ik)^l}{l!} |s_m - s|^l \right) \sigma_m(s) ds \\
&< \frac{1}{2} \sum_{l=1}^{\infty} \frac{n^l}{l!} d^{l-1} \int_{\partial D_m} |\sigma_m(s)| ds \\
&\leq \frac{1}{2} \sum_{l=1}^{\infty} \frac{n^l}{l!} d^{l-1} \cdot Ca \\
&\leq C \sum_{l=1}^{\infty} \frac{n^l}{l!} \\
&\leq C \kappa a, \quad \text{for } \lambda \leq \frac{1}{\kappa_{\max}}. \quad (2.65)
\end{align*}
\]

Define $U_m := \int_{\partial D_m} \Phi_0(s_m, s)\sigma_m(s) ds, s_m \in \partial D_m$. Then (2.64) can be written as
\[
U_m = -U^i(s_m) - \sum_{j=1}^{M} \Phi(s_m, z_j)Q_j + O(\kappa a) + O \left( (M - 1) \left( \frac{\kappa a^2}{d} + \frac{a^2}{d^2} \right) \right). \quad (2.66)
\]

For $m = 1, \ldots, M$, let $\overline{\sigma}_m \in L^2(\partial D_m)$ be the surface charge distributions which define,

- The constant potentials $\overline{U}_m$ as
\[
\int_{\partial D_m} \Phi_0(s_m, s)\overline{\sigma}_m(s) ds = \overline{U}_m := U^i(z_m) - \sum_{j=1}^{M} \Phi(z_m, z_j)Q_j, s_m \in \partial D_m. \quad (2.67)
\]

- The total charge on the surface $\partial D_m$ as
\[
\overline{Q}_m := \int_{\partial D_m} \overline{\sigma}_m(s) ds
\]

Now, we set the electrical capacitance $\overline{C}_m$ for $1 \leq m \leq M$ as
\[
\overline{C}_m := \frac{\overline{Q}_m}{\overline{U}_m}
\]

**Lemma 2.10.** We have the following estimates
\[
\|\sigma_m - \overline{\sigma}_m\|_{H^{-1}(\partial D_m)} = O \left( \kappa a + (M - 1) \left( \frac{\kappa a^2}{d} + \frac{a^2}{d^2} \right) \right), \quad (2.68)
\]
\[
Q_m - \overline{Q}_m = O \left( \kappa a^2 + (M - 1) \left( \frac{\kappa a^2}{d} + \frac{a^2}{d^2} \right) \right) \quad (2.69)
\]

where the constants appearing in $O(.)$ depend only on the Lipschitz character of $B_m$.

**Proof of Lemma 2.10.** By taking the difference between (2.66) and (2.67), we obtain
\[
U_m - \overline{U}_m = \int_{\partial D_m} \Phi_0(s_m, s) (\sigma_m - \overline{\sigma}_m) (s) ds = O(\kappa a) + O \left( (M - 1) \left( \frac{\kappa a^2}{d} + \frac{a^2}{d^2} \right) \right), \quad s_m \in \partial D_m. \quad (2.70)
\]

Indeed, by using Taylor series, we have
• \( U^i(s_m) - U^i(z_m) = O(\kappa a) \).
• \( \Phi(s_m, z_j) - \Phi(z_m, z_j) = O(\frac{\kappa a}{d} + \frac{a^2}{\delta}) \) and the estimate of \( Q_j \) given in (2.52).

In operator form we can write (2.70) as,

\[
S^*_{mm}(s_m - \sigma_m)(s_m) := \int_{\partial D_m} \Phi_0(s_m, s) (s_m - \sigma_m) (s) ds = O(\kappa a) + O\left( (M - 1) \left( \frac{\kappa a^2}{d} + \frac{a^2}{\delta^2} \right) \right), \quad s_m \in \partial D_m.
\]

Here, \( S^*_{mm} : H^{-1}(\partial D_m) \to L^2(\partial D_m) \) is the adjoint of \( S_{mm} : L^2(\partial D_m) \to H^1(\partial D_m) \). We know that,

\[
||S^*_{mm}||_{L(H^{-1}(\partial D_m), L^2(\partial D_m))} = ||S_{mm}||_{L(L^2(\partial D_m), H^1(\partial D_m))}
\]

and

\[
||S^*_{mm}^{-1}||_{L(L^2(\partial D_m), H^{-1}(\partial D_m))} = ||S_{mm}^{-1}||_{L(H^1(\partial D_m), L^2(\partial D_m))}.
\]

then from (2.36) of Proposition 2.6, we obtain \( ||S^*_{mm}||_{L(L^2(\partial D_m), H^{-1}(\partial D_m))} = O(a^{-1}) \). Hence, we get the required results in the following manner. First, we have

\[
||\sigma_m - \bar{\sigma}_m||_{H^{-1}(\partial D_m)} \leq ||S^*_{mm}^{-1}||_{L(L^2(\partial D_m), H^{-1}(\partial D_m))} \left( O(\kappa a) + O\left( (M - 1) \left( \frac{\kappa a^2}{d} + \frac{a^2}{\delta^2} \right) \right) \right)_{L^2(\partial D_m)}
\]

and second, we have

\[
||Q_m - \bar{Q}_m|| = \left| \int_{\partial D_m} (\sigma_m - \bar{\sigma}_m) (s) ds \right|
\]

= \( O\left( \kappa a + (M - 1) \left( \frac{\kappa a^2}{d} + \frac{a^2}{\delta^2} \right) \right) \).

\[
Q_m = \frac{\bar{Q}_B}{\max \{ 1 \leq m \leq M \} a} \quad \text{and} \quad \bar{Q}_m = \frac{\bar{Q}_B}{\max \{ 1 \leq m \leq M \} a}.
\]

D

**Lemma 2.11.** For every \( m, 1 \leq m \leq M \), the capacitance \( \bar{C}_m \) and the charge \( \bar{Q}_m \) are of the form;

\[
\bar{C}_m = \frac{\bar{C}_{B_m}}{\max \{ 1 \leq m \leq M \} diam(B_m)} a \quad \text{and} \quad \bar{Q}_m = \frac{\bar{Q}_{B_m}}{\max \{ 1 \leq m \leq M \} diam(B_m)} a,
\]

where \( \bar{C}_{B_m} \) and \( \bar{Q}_{B_m} \) are the capacitance and the charge of \( B_m \) respectively.

**Proof of Lemma 2.11.** Take \( 0 < \epsilon \leq 1, z \in \mathbb{R}^3 \) and write, \( D_c := B + z \subset \mathbb{R}^3 \). For \( \psi_c \in L^2(\partial D_c) \) and \( \psi \in L^2(\partial B) \), define the operators \( S^{i*} : L^2(\partial D_c) \to H^1(\partial D_c) \) and \( S^{B*}_c : L^2(\partial B) \to H^1(\partial B) \) as;

\[
S^{i*} \psi_c(x) := \int_{\partial D_c} \frac{1}{4\pi|x-y|} \psi_c(y) dy, \quad \text{and} \quad S^{B*}_c \psi_c(\xi) := \int_{\partial B} \frac{1}{4\pi|\xi-\eta|} \psi(\eta) d\eta.
\]

These two operators define the corresponding potentials \( \tilde{U}_c, \tilde{U}_B \) on the surfaces \( \partial D_c \) and \( \partial B \) w.r.t the surface charge distributions \( \psi_c \) and \( \psi \) respectively. Let these potentials be equal to some constant \( D \). Let also the total charge of these conductors \( D_c, B \) be \( \bar{Q}_c \) and \( \bar{Q}_B \), and the capacitances be \( \bar{C}_c \) and \( \bar{C}_B \) respectively. Then we can write

\[
\tilde{U}_c := S^{i*} \psi_c(x) = D, \quad \tilde{U}_B := S^{B*}_c \psi(\xi) = D, \quad \forall x \in \partial D_c, \forall \xi \in \partial B.
\]

We have by definitions, \( \bar{Q}_c = \int_{\partial D_c} \psi_c(y) dy, \quad \bar{Q}_B = \int_{\partial B} \psi(\eta) d\eta \), and \( \bar{C}_c = \frac{\bar{Q}_c}{\bar{U}_c}, \quad \bar{C}_B = \frac{\bar{Q}_B}{\bar{U}_B} \).

Observe that,

\[
D = S^{i*} \psi_c(x) = \int_{\partial D_c} \frac{1}{4\pi|x-y|} \psi_c(y) dy \quad \quad \quad D = S^{B*}_c \psi_c(\xi) = \int_{\partial B} \frac{1}{4\pi|\xi-\eta|} \psi(\eta) d\eta
\]
As we have
\[ \psi(x) = \frac{1}{\epsilon} \psi(y) \]
which gives us,
\[ \psi(x) = \frac{1}{\epsilon} \psi(y) \]
Hence, \( \psi_x = \frac{1}{\epsilon} \psi \) and \( \psi = \epsilon \psi_x \). Now we have,
\[
Q_x = \int_{\partial D_x} \psi_x(y)dy = \int_{\partial D_x} \frac{1}{\epsilon} \psi(y)dy, \]
\[
= \int_{\partial B} \frac{1}{\epsilon} \psi(\eta + z) \epsilon^2 d\eta = \epsilon \int_{\partial B} \psi(\eta + z) d\eta, \]
\[
= \epsilon \int_{\partial B} \psi(\eta) d\eta = \epsilon \int_{\partial B} \psi(\eta) d\eta, \]
\[
= \epsilon \int_{\partial B} \psi(\eta) d\eta \]
which gives us,
\[
C_x = \frac{Q_x}{U_x} = \frac{Q_B}{U_B} = \epsilon \frac{Q_B}{U_B} = \epsilon C_B. \]
As we have \( D_m = \epsilon B_m + z_m \) and \( a = \max_{1 \leq m \leq M} \text{diam}(B_m) \), we obtain
\[
Q_m = \epsilon Q_{B_m} = \frac{Q_{B_m}}{\max_{1 \leq m \leq M} \text{diam}(B_m)} a \text{ and } C_m = \epsilon C_{B_m} = \frac{C_{B_m}}{\max_{1 \leq m \leq M} \text{diam}(B_m)} a. \]

**Proposition 2.12.** For \( m = 1, 2, \ldots, M \), the total charge \( Q_m \) on each surface \( \partial D_m \) of the small scatterer \( D_m \) can be calculated from the algebraic system
\[
\frac{Q_m}{\epsilon C_m} = -U^1(z_m) - \sum_{j=1}^{M} \frac{\partial}{\partial z_j} \Phi(z_m, z_j) \frac{Q_j}{C_j} + O \left( (M-1) \frac{\kappa a^2}{d} + (M-1)^2 \left( \frac{\kappa a^3}{d^2} + \frac{a^3}{d^4} \right) \right). \tag{2.72}
\]

**Proof of Proposition 2.12.** We can rewrite (2.67) as
\[
\frac{Q_m}{\epsilon C_m} = -U^1(z_m) - \sum_{j=1}^{M} \Phi(z_m, z_j) Q_j
\]
\[
= -U^1(z_m) - \sum_{j=1}^{M} \Phi(z_m, z_j) Q_j - \sum_{j=1}^{M} \Phi(z_m, z_j) (Q_j - Q_j)
\]
\[
= -U^1(z_m) - \sum_{j=1}^{M} \Phi(z_m, z_j) Q_j + O \left( (M-1) \frac{\kappa a^2}{d} + (M-1)^2 \left( \frac{\kappa a^3}{d^2} + \frac{a^3}{d^4} \right) \right),
\]
where we used (2.69) and the fact \( \Phi(z_m, z_j) = O \left( \frac{1}{\lambda} \right) \).
2.2. Using DLP representation.

2.2.1. The representation via DLP.

**Proposition 2.13.** For \( m = 1, 2, \ldots, M \), there exists \( \sigma_m \in H^r(\partial D_m) \), \( r \in [0, 1] \) such that the solution of the problem (1.1-1.3) is of the form

\[
U^i(x) = U^i(x) + \sum_{m=1}^{M} \int_{\partial D_m} \frac{\partial \Phi(x, s)}{\partial n_m(s)} \sigma_m(s) ds, \quad x \in \mathbb{R}^3 \cup \left( \bigcup_{m=1}^{M} D_m \right),
\]  

(2.73)

\( \Phi(x, y) := \frac{e^{i|x-y|}}{4\pi|x-y|} \), for all \( x, y \in \mathbb{R}^3 \). Here \( n_m \) is the outward unit normal vector of \( \partial D_m \).

**Proof of Proposition 2.13.** We look for the solution of the problem (1.1-1.3) of the form (2.73), then from the Dirichlet condition (1.2), we obtain

\[
\frac{\sigma_m(s)}{2} + \int_{\partial D_m} \frac{\partial \Phi(s_j, s)}{\partial n_j(s)} \sigma_j(s) ds + \sum_{m=1}^{M} \int_{\partial D_m} \frac{\partial \Phi(s_j, s)}{\partial n_m(s)} \sigma_m(s) ds = -U^i(s_j), \quad \forall s_j \in \partial D_j, \quad j = 1, \ldots, M.
\]  

(2.74)

One can write it in compact form as \( (\frac{1}{2}I + DL + DK)\sigma = -U^{Jn} \) with \( I := (I_{mj})_{m,j=1}^{M} \), \( DL := (D_{lj})_{m,j=1}^{M} \) and \( DK := (DK_{mj})_{m,j=1}^{M} \), where

\[
I_{mj} = \begin{cases} 1 & m = j, \\ 0 & \text{else}, \end{cases}, \quad DL_{mj} = \begin{cases} D_{lj} & m = j, \\ 0 & \text{else}, \end{cases}, \quad DK_{mj} = \begin{cases} D_{lj} & m \neq j, \\ 0 & \text{else}. \end{cases}
\]  

(2.75)

\( U^{Jn} = (U^i, \ldots, U^i)^T \) and \( \sigma = (\sigma_1, \ldots, \sigma_M)^T \). Here, for the indices \( m \) and \( j \) fixed, \( D_{mj} \) is the integral operator acting as

\[
D_{mj}(\sigma_j)(t) := \int_{\partial D_j} \frac{\partial \Phi(t, s)}{\partial n_j(s)} \sigma_j(s) ds, \quad t \in \partial D_m.
\]  

(2.76)

The operator \( \frac{1}{2}I + DL_{mm} : H^r(\partial D_m) \to H^r(\partial D_m) \) is Fredholm with zero index and for \( m \neq j \), \( D_{mj} : H^r(\partial D_j) \to H^r(\partial D_m) \) is compact for \( 0 \leq r \leq 1 \), when \( \partial D_m \) has a Lipschitz regularity, see [19].

So, \( (\frac{1}{2}I + DL + DK) : \prod_{m=1}^{M} H^r(\partial D_m) \to \prod_{m=1}^{M} H^r(\partial D_m) \) is Fredholm with zero index. We induce the product of spaces by the maximum of the norms of the space. To show that \( (\frac{1}{2}I + DL + DK) \sigma = 0 \) implies \( \sigma = 0 \).

Write,

\[
\tilde{U}(x) = \frac{M}{m=1} \int_{\partial D_m} \frac{\partial \Phi(s_j, s)}{\partial n_m(s)} \sigma_m(s) ds, \quad x \in \mathbb{R}^3 \cup \left( \bigcup_{m=1}^{M} D_m \right)
\]

and

\[
\tilde{U}(x) = \sum_{m=1}^{M} \int_{\partial D_m} \frac{\partial \Phi(s_j, s)}{\partial n_m(s)} \sigma_m(s) ds, \quad x \in \bigcup_{m=1}^{M} D_m.
\]

Then \( \tilde{U} \) satisfies \( \Delta \tilde{U} + \kappa^2 \tilde{U} = 0 \) for \( x \in \mathbb{R}^3 \cup \left( \bigcup_{m=1}^{M} D_m \right) \), with S.R.C and \( \tilde{U}(x) = 0 \) on \( \bigcup_{m=1}^{M} \partial D_m \).

Similarly, \( \tilde{U} \) satisfies \( \Delta \tilde{U} + \kappa^2 \tilde{U} = 0 \) for \( x \in \bigcup_{m=1}^{M} D_m \) with \( \tilde{U}(x) = 0 \) on \( \bigcup_{m=1}^{M} \partial D_m \). By the Jump relations, we have

\[
\tilde{U}(s) = 0 \Rightarrow (K\sigma_m)(s) + \frac{\sigma_m(s)}{2} + \sum_{j \neq m} D_{mj}(\sigma_j)(s) = 0
\]  

(2.77)

\[\]  

\[\]
and
\[
\tilde{U}(s) = 0 \implies (K\sigma_m)(s) - \frac{\sigma_m(s)}{2} + \sum_{j \neq m} D_{mj}(\sigma_j)(s) = 0
\]  
(2.78)
for \( s \in \partial D_m \) and for \( m = 1, \ldots, M \). Here we recall that
\[
(K\sigma_m)(s) = \int_{\partial D_m} \frac{\partial \Phi(s, t)}{\partial \nu_m(t)} \sigma_m(t) dt, \quad \text{for } m = 1, \ldots, M.
\]
Difference between (2.77) and (2.78) provide us, \( \sigma_m = 0 \) for all \( m \).
We conclude then that \( \frac{1}{2}I + DL + DK = \frac{1}{2}I + D : \prod_{m=1}^{M} H^r(\partial D_m) \to \prod_{m=1}^{M} H^r(\partial D_m) \) is invertible.

\[ \square \]

2.2.2. An appropriate estimate of the densities \( \sigma_m, m = 1, \ldots, M \). From the above theorem, we have the following representation of \( \sigma \):
\[
\sigma = (\frac{1}{2}I + DL + DK)^{-1}U^{In}
\]
\[
= (\frac{1}{2}I + DL)^{-1}(I + (\frac{1}{2}I + DL)^{-1}DK)^{-1}U^{In}
\]
\[
= (\frac{1}{2}I + DL)^{-1} \sum_{l=0}^{\infty} \left(-\left(\frac{1}{2}I + DL\right)^{-1}DK\right)^l U^{In}, \quad \text{if } \left\| \frac{1}{2}I + DL)^{-1}DK \right\| < 1. \]  
(2.79)

The operator \( \frac{1}{2}I + DL \) is invertible since it is Fredholm of index zero and injective. This implies that
\[
||\sigma|| \leq \frac{||\left(\frac{1}{2}I + DL\right)^{-1}||}{1 - ||(\frac{1}{2}I + DL)^{-1}|| ||DK||} ||U^{In}||. \]  
(2.80)

We use the following notations
\[
||DK|| := ||DK||_{L^2(\prod_{m=1}^{M} L^2(\partial D_m))}, \quad \equiv \prod_{m=1}^{M} \max_{j=1}^{M} ||D_{mj}||_{L^2(\partial D_m)} \]
\[
\equiv M \max_{m=1}^{M} \sum_{j=1}^{M} ||D_{mj}||_{L^2(\partial D_j)}, \quad \text{(2.81)}
\]

\[
\left\| \left(\frac{1}{2}I + DL\right)^{-1} \right\| := \left\| \left(\frac{1}{2}I + DL\right)^{-1} \right\|_{L^2(\prod_{m=1}^{M} L^2(\partial D_m))}, \quad \equiv M \max_{m=1}^{M} \sum_{j=1}^{M} \left\| \left(\frac{1}{2}I + DL\right)^{-1} \right\|_{L^2(\partial D_m)}, \quad \text{(2.82)}
\]

\[
||\sigma|| := ||\sigma||_{L^2(\partial D_m)}, \quad \equiv \max_{m=1}^{M} ||\sigma_m||_{L^2(\partial D_m)} \quad \text{(2.83)}
\]

and \( ||U^{In}|| := ||U^{In}||_{\prod_{m=1}^{M} L^2(\partial D_m)}, \quad \equiv \max_{m=1}^{M} ||U^I||_{L^2(\partial D_m)} \)  
(2.84)
In the following proposition, we provide conditions under which \( \|L^{-1}\| \|K\| < 1 \) and then estimate \( \|\sigma\| \) via \( (2.80) \).

**Proposition 2.14.** There exists a constant \( \epsilon \) depending only on the Lipschitz character of \( B_m, m = 1, \ldots, M, d_{\text{max}} \) and \( \kappa_{\text{max}} \) such that if \( \sqrt{M} - 1 \epsilon < \epsilon d \), then the \( L^2 \)-norm of densities \( \sigma_m \) appearing in the solution \( (2.73) \) of the problem \( (1.1-1.3) \) are bounded by a uniform constant times \( \epsilon \).

Here as well, we divide the proof of Proposition 2.14 into two steps. In the first step, we assume we have a single obstacle and then in the second step we deal with the multiple obstacle case.

### 2.2.2.1. The case of a single obstacle.

Let us consider a single obstacle \( D_e := \epsilon B + z \). Then define the operator \( \mathcal{D}_{D_e} : L^2(\partial D_e) \to L^2(\partial D_e) \) by

\[
(\mathcal{D}_{D_e} \psi)(s) := \int_{\partial D_e} \frac{\partial \Phi(s,t)}{\partial \nu(t)} \psi(t) dt.
\]

(2.85)

Following the arguments in the proof of Proposition 2.13, the integral operator \( \frac{1}{2} I + \mathcal{D}_{D_e} : L^2(\partial D_e) \to L^2(\partial D_e) \) is invertible. If we consider the problem \( (1.1-1.3) \) in \( \mathbb{R}^3 \setminus D_e \), we obtain

\[
\sigma = \left( \frac{1}{2} I + \mathcal{D}_{D_e} \right)^{-1} U^i, \quad \text{where} \quad DL + DK = \mathcal{D}_{D_e}
\]

and then

\[
\|\sigma\|_{L^2(\partial D_e)} \leq \left\| \left( \frac{1}{2} I + \mathcal{D}_{D_e} \right)^{-1} \|_{\mathcal{L}(L^2(\partial D_e), L^2(\partial D_e))} \right\| U^i \|_{L^2(\partial D_e)}. \tag{2.86}
\]

**Lemma 2.15.** Let \( \phi, \psi \in L^2(\partial D_e) \). Then,

\[
\mathcal{D}_{D_e} \psi = (\mathcal{D}_{D_e}^2 \psi)^\vee.
\]

(2.87)

\[
\left( \frac{1}{2} I + \mathcal{D}_{D_e} \right) \psi = \left( \frac{1}{2} I + \mathcal{D}_{B} \right)^\vee \psi,
\]

(2.88)

\[
\left( \frac{1}{2} I + \mathcal{D}_{D_e} \right)^{-1} = \left( \frac{1}{2} I + \mathcal{D}_{B} \right)^{-1} \phi.
\]

(2.89)

\[
\left\| \left( \frac{1}{2} I + \mathcal{D}_{D_e} \right)^{-1} \|_{\mathcal{L}(L^2(\partial D_e), L^2(\partial D_e))} = \left\| \left( \frac{1}{2} I + \mathcal{D}_{B} \right)^{-1} \|_{\mathcal{L}(L^2(\partial B), L^2(\partial B))} \right. \tag{2.90}
\]

and

\[
\left\| \left( \frac{1}{2} I + \mathcal{D}_{D_e} \right)^{-1} \|_{\mathcal{L}(H^1(\partial D_e), H^1(\partial D_e))} \leq \epsilon^{-1} \left\| \left( \frac{1}{2} I + \mathcal{D}_{B} \right)^{-1} \|_{\mathcal{L}(H^1(\partial B), H^1(\partial B))} \right. \tag{2.91}
\]

with \( \mathcal{D}_{B} \psi(\xi) := \int_{\partial B} \frac{\partial \Phi(\xi, \eta)}{\partial \nu(\eta)} \psi(\eta) d\eta \) and \( \Phi(\xi, \eta) := \frac{e^{i\xi(\eta)} - e^{i\xi(\eta)}}{4\pi |\xi - \eta|} \).

**Proof of Lemma 2.15.**

- We have,

\[
\mathcal{D}_{D_e} \psi(s) = \int_{\partial D_e} \frac{\partial \Phi(s,t)}{\partial \nu(t)} \psi(t) dt
\]

\[
= \int_{\partial D_e} \frac{e^{i\xi(s-t)}}{4\pi |s-t|} \left[ \frac{1}{s-t} - ik \frac{s-t}{|s-t|} \right] \cdot \nu(t) \psi(t) dt
\]

\[
= \int_{\partial B} \frac{e^{i\xi(s-t)}}{4\pi \epsilon |\xi - \eta|} \left[ \frac{1}{\epsilon(\xi - \eta)} - ik \frac{\xi - \eta}{|\xi - \eta|} \right] \cdot \nu(\eta) \psi(\epsilon \eta + z) \epsilon^2 d\eta
\]
The following equalities

\[
\psi(s) = \frac{1}{2} \psi(s) + \mathcal{D}_s \psi(s) = \frac{1}{2} \hat{\psi}(\xi) + \mathcal{D}_s \hat{\psi}(\xi) = \left( \frac{1}{2} I + \mathcal{D}_s \right) \hat{\psi}(\xi).
\]

• The following equalities

\[
\left( \frac{1}{2} I + \mathcal{D}_s \right) \left( \left( \frac{1}{2} I + \mathcal{D}_s \right)^{-1} \phi \right)^\nu = \left( \frac{1}{2} I + \mathcal{D}_s \right) \left( \left( \frac{1}{2} I + \mathcal{D}_s \right)^{-1} \phi \right)^\nu = \phi^\nu = \phi,
\]

provide us (2.89).

• We have from the estimate,

\[
\left\| \left( \frac{1}{2} I + \mathcal{D}_s \right)^{-1} \right\|_{\mathcal{L}(L^2(\partial D_s), L^2(\partial D_s))} := \sup_{\phi \neq 0 \in L^2(\partial D_s)} \frac{\left\| \left( \frac{1}{2} I + \mathcal{D}_s \right)^{-1} \phi \right\|_{L^2(\partial D_s)}}{\left\| \phi \right\|_{L^2(\partial D_s)}} \leq \epsilon \left\| \left( \frac{1}{2} I + \mathcal{D}_s \right)^{-1} \phi \right\|^\vee_{L^2(\partial B)} \left\| \mathcal{D}_s \phi \right\|_{L^2(\partial B)}/\epsilon \left\| \phi \right\|_{L^2(\partial B)} \leq \sup_{\phi \neq 0 \in L^2(\partial D_s)} \frac{\left\| \left( \frac{1}{2} I + \mathcal{D}_s \right)^{-1} \phi \right\|_{L^2(\partial B)}}{\left\| \phi \right\|_{L^2(\partial B)}} \leq \left\| \left( \frac{1}{2} I + \mathcal{D}_s \right)^{-1} \phi \right\|_{L^2(\partial B)}/\epsilon \left\| \phi \right\|_{L^2(\partial B)} \leq \left\| \left( \frac{1}{2} I + \mathcal{D}_s \right)^{-1} \phi \right\|_{L^2(\partial B)} \leq \epsilon \left\| \mathcal{D}_s \phi \right\|_{L^2(\partial B)}/\epsilon \left\| \phi \right\|_{L^2(\partial B)}.
\]

It provides us (2.90). By proceeding in the similar manner we can obtain (2.91) as mentioned below.
Lemma 2.16. The operator norm of the inverse of $\frac{1}{2}I + D_{D_1} : L^2(\partial D_1) \to L^2(\partial D_1)$ defined by $D_{D_1}\psi(s) := \int_{\partial D_1} \frac{\partial \Phi(s,t)}{\partial n(t)} \psi(t) dt$ in (2.85), is bounded by a constant, i.e.

$$\left\| \left( \frac{1}{2}I + D_{D_1} \right)^{-1} \right\|_{\mathcal{L}(L^2(\partial D_1),L^2(\partial D_1))} \leq \mathcal{C}_6,$$  \hspace{1cm} (2.92)

with $\mathcal{C}_6 := \frac{4\pi \left\| \left( \frac{1}{2}I + D_{D_1}^h \right)^{-1} \right\|_{\mathcal{L}(L^2(\partial B),L^2(\partial B))}}{4\pi - \kappa^2 |\partial B| \left\| \left( \frac{1}{2}I + D_{D_1}^h \right)^{-1} \right\|_{\mathcal{L}(L^2(\partial B),L^2(\partial B))}}$. Here, $D_{D_1}^h : L^2(\partial B) \to L^2(\partial B)$ is the double layer potential with the wave number zero.

Here we should mention that if $\epsilon^2 \leq \frac{\pi}{|\partial B| \left\| \left( \frac{1}{2}I + D_{D_1}^h \right)^{-1} \right\|_{\mathcal{L}(L^2(\partial B),L^2(\partial B))}}$, then $\mathcal{C}_6$ is bounded by

$$\frac{4\pi}{4\pi - \kappa^2 |\partial B|^2} \left\| \left( \frac{1}{2}I + D_{D_1}^h \right)^{-1} \right\|_{\mathcal{L}(L^2(\partial B),L^2(\partial B))},$$

a universal constant depending only on $\partial B$ through its Lipschitz character, see Remark 2.2.2.

Proof of Lemma 2.16. To estimate the operator norm of $\left( \frac{1}{2}I + D_{D_1} \right)^{-1}$ we decompose $D_{D_1} =: D_{D_1}^c + D_{D_1}^d$ into two parts $D_{D_1}^c$ (independent of $\kappa$) and $D_{D_1}^d$ (dependent of $\kappa$) given by

$$D_{D_1}^c \psi(s) := \int_{\partial D_1} \left( \frac{\partial}{\partial n(t)} \Phi_0(s,t) \right) \psi(t) dt,$$

$$D_{D_1}^d \psi(s) := \int_{\partial D_1} \left( \frac{\partial}{\partial n(t)} [\Phi(s,t) - \Phi_0(s,t)] \right) \psi(t) dt.$$  \hspace{1cm} (2.93), (2.94)

With this definition, $\frac{1}{2}I + D_{D_1}^c : L^2(\partial D_1) \to L^2(\partial D_1)$ is invertible, see [19]. Hence, $\frac{1}{2}I + D_{D_1} = (\frac{1}{2}I + D_{D_1}^c) \left( I + \left( \frac{1}{2}I + D_{D_1}^c \right)^{-1} D_{D_1}^d \right)$ and so

$$\left\| \left( \frac{1}{2}I + D_{D_1} \right)^{-1} \right\|_{\mathcal{L}(L^2(\partial D_1),L^2(\partial D_1))} \leq \left\| \left( I + \left( \frac{1}{2}I + D_{D_1}^c \right)^{-1} D_{D_1}^d \right)^{-1} \right\|_{\mathcal{L}(L^2(\partial D_1),L^2(\partial D_1))}$$

$$\times \left\| \left( \frac{1}{2}I + D_{D_1}^c \right)^{-1} \right\|_{\mathcal{L}(L^2(\partial D_1),L^2(\partial D_1))}.$$  \hspace{1cm} (2.95)

So, to estimate the operator norm of $\left( \frac{1}{2}I + D_{D_1} \right)^{-1}$ one need to estimate the operator norm of $\left( I + \left( \frac{1}{2}I + D_{D_1}^c \right)^{-1} D_{D_1}^d \right)^{-1}$, in particular one need to have the knowledge about the operator norms of $\left( \frac{1}{2}I + D_{D_1}^c \right)^{-1}$ and $D_{D_1}^d$ to apply the Neumann series. For that purpose, we can estimate the operator norm of $\left( \frac{1}{2}I + D_{D_1}^c \right)^{-1}$ from (2.90) by

$$\left\| \left( \frac{1}{2}I + D_{D_1}^c \right)^{-1} \right\|_{\mathcal{L}(L^2(\partial D_1),L^2(\partial D_1))} = \left\| \left( \frac{1}{2}I + D_{D_1}^h \right)^{-1} \right\|_{\mathcal{L}(L^2(\partial B),L^2(\partial B))}.$$  \hspace{1cm} (2.96)

Here $D_{D_1}^h \hat{\psi}(\xi) := \int_{\partial B} \frac{\delta_{k(\xi,s)}}{4\pi |\xi - s|} \nu(\eta) \hat{\psi}(\eta) d\eta$. From the definition of the operator $D_{D_1}^d$, in (2.94), by performing the similar calculation as made in (2.27), we deduce that

$$D_{D_1}^d \psi(s) = \int_{\partial D_1} \left( \frac{\partial}{\partial n(t)} \frac{e^{ik|s-t|} - 1}{4\pi |s-t|} \right) \psi(t) dt.$$
\[
\begin{align*}
\int_{\partial D_i} \left( \frac{e^{ik|s-t|}}{4\pi |s-t|} \left[ \frac{1}{|s-t|} - \frac{1}{4\pi |s-t|^3} \right] - \frac{s-t}{|s-t|^3} \right) \cdot \nu(t) \psi(t) dt \\
= \frac{k^2 \epsilon^2}{4\pi} \int_{\partial B} \left( \sum_{l=1}^{\infty} \left( \frac{1}{l} - \frac{1}{l+1} \right) (i\kappa)^{l-1} |\xi - \eta|^{l-1} \right) \frac{\xi - \eta}{|\xi - \eta|^3} \cdot \nu(\eta) \hat{\psi}(\eta) d\eta.
\end{align*}
\]

(2.97)

Now, by performing the similar calculations made in (2.30), (2.97) will direct us to calculate the below

\[
\left| D_{D_i}^{D_{D_j}} \psi(s) \right| \leq \frac{k^2 \epsilon^2}{4\pi} \left( \sum_{l=1}^{\infty} \left( \frac{1}{l} - \frac{1}{l+1} \right) (i\kappa)^{l-1} |\xi - \eta|^{l-1} \right) \left\| \hat{\psi} \right\|_{L^2(\partial B)}^{\frac{1}{m}} \leq \frac{k^2 \epsilon^2}{4\pi} |\partial B|^{\frac{1}{2}} \left\| \hat{\psi} \right\|_{L^2(\partial B)}^{\frac{1}{m}}
\]

(2.98)

For set \( \hat{C}_1 := \frac{|\partial B|^\frac{1}{2}}{4\pi} \). From (2.98), by doing the similar calculations used to derive (2.99), we obtain

\[
\left| D_{D_i}^{D_{D_j}} \psi \right|_{L^2(\partial D_i)} \leq \hat{C}_1 k^2 \epsilon^3 |\partial B|^{\frac{1}{2}} \left\| \hat{\psi} \right\|_{L^2(\partial B)}.
\]

(2.99)

We estimate norm of the operator \( D_{D_i}^{D_{D_j}} \) as

\[
\left\| D_{D_i}^{D_{D_j}} \right\|_{L^2(\partial D_i), L^2(\partial D_j)} = \sup_{\psi(\not\in L^2(\partial D_i))} \frac{|D_{D_i}^{D_{D_j}} \psi|_{L^2(\partial D_j)}}{\left\| \psi \right\|_{L^2(\partial D_i)}} \leq \hat{C}_1 k^2 \epsilon^3 |\partial B|^{\frac{1}{2}} \left\| \hat{\psi} \right\|_{L^2(\partial B)}
\]

(2.100)

Hence, we get

\[
\left\| \left( \frac{1}{2} I + D_{D_i}^{D_{D_j}} \right)^{-1} D_{D_i}^{D_{D_j}} \right\|_{L^2(\partial D_i), L^2(\partial D_j)} \leq \hat{C}_1 k^2 \epsilon^3 |\partial B|^{\frac{1}{2}}
\]

(2.101)

where \( \hat{C}_2 := \hat{C}_1 |\partial B|^{\frac{1}{2}} \left\| \left( \frac{1}{2} I + D_{D_i}^{D_{D_j}} \right)^{-1} \right\|_{L^2(\partial D_i), L^2(\partial D_j)} \leq \frac{|\partial B|}{4\pi} \left\| \left( \frac{1}{2} I + D_{D_i}^{D_{D_j}} \right)^{-1} \right\|_{L^2(\partial D_i), L^2(\partial D_j)} \) Assuming \( \epsilon \) to satisfy the condition \( \epsilon < \frac{1}{\sqrt{\hat{C}_2 k^2 \epsilon^3}} \), then \( \left\| \left( \frac{1}{2} I + D_{D_i}^{D_{D_j}} \right)^{-1} D_{D_i}^{D_{D_j}} \right\|_{L^2(\partial D_i), L^2(\partial D_j)} < 1 \) and hence by using the Neumann series we obtain the following

\[
\left\| \left( I + \left( \frac{1}{2} I + D_{D_i}^{D_{D_j}} \right)^{-1} D_{D_i}^{D_{D_j}} \right)^{-1} \right\|_{L^2(\partial D_i), L^2(\partial D_j)} \leq \frac{1}{1 - \left\| \left( \frac{1}{2} I + D_{D_i}^{D_{D_j}} \right)^{-1} D_{D_i}^{D_{D_j}} \right\|_{L^2(\partial D_i), L^2(\partial D_j)}}
\]

(2.102)

By substituting the above and (2.96) in (2.95), we obtain the required result (2.92).

\[\text{ Proof.} \]

\textbf{2.2.2. The multiple obstacle case.}

\textbf{Proposition 2.17.} For \( m, j = 1, 2, \ldots, M \), the operator \( D_{m_j} : L^2(\partial D_j) \rightarrow L^2(\partial D_m) \) defined by

\[
D_{m_j}(\sigma_j)(s_m) := \int_{\partial D_j} \frac{\partial \Phi(s_m, t)}{\partial n_j(t)} \sigma_j(t) dt, \quad s_m \in \partial D_m
\]

(2.102)

enjoys the following estimates,
For $j = m$,

$$
\left\| \left( \frac{1}{2} I + D_{mm} \right)^{-1} \right\|_{L^2(\partial D_m), L^2(\partial D_m)} \leq \hat{C}_{6m},
$$

where $\hat{C}_{6m} := \frac{4\pi}{4\pi - \kappa^2 \epsilon^2 |\partial B_m|} \left| \left( \frac{1}{2} I + D_{Bm} \right)^{-1} \right|_{L^2(\partial B_m), L^2(\partial B_m)}$.

For $j \neq m$,

$$
\left\| D_{mj} \right\|_{L^2(\partial D_j), L^2(\partial D_m)} \leq \frac{1}{4\pi} \left( \frac{\kappa}{d} + \frac{1}{d^2} \right) |\partial B_j| \epsilon^2,
$$

where $|\partial B| := \max_j |\partial B_j|$.

**Proof of Proposition 2.17.** The estimate (2.103) is nothing but (2.92) of Lemma 2.16, replacing $B$ by $B_m$, $z$ by $z_m$ and $D\epsilon$ by $D_m$ respectively. It remains to prove the estimate (2.104). We have

$$
\left\| D_{mj} \right\|_{L^2(\partial D_j), L^2(\partial D_m)} \leq \sup \frac{\left\| D_{mj} \right\|_{L^2(\partial D_j)}}{\left\| \psi \right\|_{L^2(\partial D_j)}}
$$

Let $\psi \in L^2(\partial D_j)$ then for $s \in \partial D_m$, we have

$$
|D_{mj} \psi(s)| = \left| \int_{\partial D_j} \frac{\partial \Phi(s, t)}{\partial v_j(t)} \psi(t) dt \right|
= \left| \int_{\partial D_j} \nabla_t \Phi(s, t) \cdot v_j(t) \psi(t) dt \right|
\leq \int_{\partial D_j} |\nabla_t \Phi(s, t)| \left| \psi(t) \right| dt
\leq \frac{1}{4\pi} \left( \frac{\kappa}{d_{mj}} + \frac{1}{d^2_{mj}} \right) \epsilon |\partial B_j| \frac{1}{2} \left\| \psi \right\|_{L^2(\partial D_j)},
$$

Here we used the similar calculations made in (2.41). From (2.106), by proceeding further in the way of (2.42), we get

$$
\left\| D_{mj} \psi \right\|_{L^2(\partial D_m)} \leq \frac{1}{4\pi} \left( \frac{\kappa}{d_{mj}} + \frac{1}{d^2_{mj}} \right) \epsilon^2 |\partial B_j| \frac{1}{2} \left\| \psi \right\|_{L^2(\partial D_j)},
$$

(2.107)

Substitution of (2.107) in (2.105) gives us

$$
\left\| D_{mj} \right\|_{L^2(\partial D_j), L^2(\partial D_m)} \leq \frac{1}{4\pi} \left( \frac{\kappa}{d_{mj}} + \frac{1}{d^2_{mj}} \right) \epsilon^2 |\partial B_j| \frac{1}{2} |\partial B_m| \frac{1}{2} \leq \frac{1}{4\pi} \left( \frac{\kappa}{d} + \frac{1}{d^2} \right) |\partial B| \epsilon^2.
$$

End of the proof of Proposition 2.14. By substituting (2.103) in (2.82) and (2.104) in (2.81), we obtain

$$
\left\| DK \right\| \equiv \max_{m=1}^{M} \sum_{j=1}^{M} \left\| D_{mj} \right\|_{L^2(\partial D_j), L^2(\partial D_m)} \leq \frac{M - 1}{4\pi} \left( \frac{\kappa}{d} + \frac{1}{d^2} \right) |\partial B| \epsilon^2
$$

(2.108)
and
\[
\left\| \left( \frac{1}{2} I + DL \right)^{-1} \right\| \equiv \max_{m=1}^{M} \left\| \left( \frac{1}{2} I + DL_{mm} \right)^{-1} \right\|_{L(L^2(\partial D_m), L^2(\partial D_m))} 
\]
\[
\equiv \max_{m=1}^{M} \hat{C}_{6m}.
\]

Hence, (2.108) and (2.109) jointly provide
\[
\left\| \left( \frac{1}{2} I + DL \right)^{-1} \right\| \|DK\| \leq \frac{M - 1}{4\pi} \left( \max_{m=1}^{M} \hat{C}_{6m} \right) |\partial B| \left( \frac{\kappa}{d} + \frac{1}{d^2} \right) \epsilon^2, \quad (2.110)
\]

By imposing the condition \( \left\| \left( \frac{1}{2} I + DL \right)^{-1} \right\| \|DK\| < 1 \), we get the following from (2.80) and (2.83-2.84);
\[
\|\sigma_m\|_{L^2(\partial D_m)} \leq \|\sigma\| \quad \leq \quad \frac{\left\| \left( \frac{1}{2} I + DL \right)^{-1} \right\|}{1 - \left\| \left( \frac{1}{2} I + DL \right)^{-1} \right\| \|DK\|} \|U^m\| 
\]
\[
\leq \hat{C}_p \left( \frac{1}{2} I + DL \right)^{-1} \left\| U^m \right\|_{L^2(\partial D_m)} \left( \hat{C}_p \geq \frac{1}{1 - \hat{C}_s} \right) 
\]
\[
(2.109) \leq \hat{C}_p \left( \frac{1}{2} I + DL \right)^{-1} \|U^m\|_{L^2(\partial D_m)} \left( \hat{C} := \hat{C}_p \max_{m=1}^{M} \hat{C}_{6m} \right), \quad (2.111)
\]

for all \( m \in \{1, 2, \ldots, M\} \). But,
\[
\|U^m\|_{L^2(\partial D_m)} = \epsilon |\partial B_m|^{\frac{1}{2}}, \forall m = 1, 2, \ldots, M. \quad (2.112)
\]

Now by substituting (2.112) in (2.111), for each \( m = 1, \ldots, M \), we obtain
\[
\|\sigma_m\|_{L^2(\partial D_m)} \leq \hat{C}(\kappa) \epsilon, \quad (2.113)
\]
where \( \hat{C}(\kappa) := \hat{C} |\partial B|^{\frac{1}{2}} \).

The condition \( \left\| \left( \frac{1}{2} I + DL \right)^{-1} \right\| \|DK\| < 1 \) is satisfied if
\[
\frac{M - 1}{4\pi} |\partial B| \left( \frac{\kappa}{d} + \frac{1}{d^2} \right) \left( \max_{m=1}^{M} \hat{C}_{6m} \right) \epsilon^2 < 1. \quad (2.114)
\]

Since \( \kappa d < \check{\epsilon} \) for \( \check{\epsilon} := \kappa_{\max} d_{\max} \), then (2.114) reads as \((M - 1)\epsilon^2 < cd^2\), where we set \( c := \left( \frac{\kappa_{\max} d_{\max}}{4\pi} |\partial B| \max_{m=1}^{M} \hat{C}_{6m} \right)^{-1} \) and \( \check{\epsilon} := \sqrt{c} \) will serve our purpose in Proposition 2.14 and hence in Theorem 1.2 for the case \( \sqrt{M} - 1 \epsilon < cd \).

Remark here that if we do not have the condition (1.7), then (2.114) holds if \((M - 1)\epsilon < cd\) with \( c := \frac{1}{2} \left( \frac{\kappa_{\max} d_{\max}}{4\pi} \max_{m=1}^{M} \hat{C}_{6m} \right)^{-1} \) (for \( M \) large enough). Then \( c_2 := c \max_{1 \leq m \leq M} diam(B_m) \) will serve our purpose in Remark (1.3).

For \( m = 1, 2, \ldots, M \), let \( U^m \sigma \) be the solution of the problem
\[
\begin{aligned}
(U^m + \kappa^2)U^m \sigma &= 0 \quad \text{in } D_m, \\
U^m \sigma &= \sigma_m \quad \text{on } \partial D_m.
\end{aligned} \quad (2.115)
\]
The function $\sigma_m$ is in $H^1(\partial D_m)$, see Proposition 2.13. Hence $U^{\sigma_m} \in H^2(D_m)$ and then $\frac{\partial U^{\sigma_m}}{\partial n_m} \big|_{\partial D_m} \in L^2(\partial D_m)$. From Proposition 2.13, the solution of the problem (1.1-1.3) has the form

$$U^i(x) = U^i(x) + \sum_{m=1}^M \int_{\partial D_m} \frac{\partial \Phi(x,s)}{\partial n_m(s)} \sigma_m(s) ds, \ x \in \mathbb{R}^3 \setminus \left( \bigcup_{m=1}^M \bar{D}_m \right). \quad (2.116)$$

It can be written in terms of single layer potential using Gauss theorem as

$$U^i(x) = U^i(x) + \sum_{m=1}^M \int_{\partial D_m} \Phi(x,s) \frac{\partial U^{\sigma_m}(s)}{\partial n_m(s)} ds, \ x \in \mathbb{R}^3 \setminus \left( \bigcup_{m=1}^M \bar{D}_m \right). \quad (2.117)$$

Indeed,

$$\int_{\partial D_m} \frac{\partial \Phi(x,s)}{\partial n_m(s)} \sigma_m(s) ds = \int_{\partial D_m} \Phi(x,s) \frac{\partial U^{\sigma_m}(s)}{\partial n_m(s)} ds + \int_{D_m} [U^{\sigma_m}(s) \Delta \Phi(x,s) - \Phi(x,s) \Delta U^{\sigma_m}(s)] ds.$$ 

**Lemma 2.18.** For $m = 1, 2, \ldots, M$, $U^{\sigma_m}$, the solutions of the problem (2.115) satisfy the estimate

$$\left\| \frac{\partial U^{\sigma_m}(s)}{\partial n_m(s)} \right\|_{H^{-1}(\partial D_m)} \leq C_7, \quad (2.118)$$

for some constant $C_7$ depending on the Lipschitz character of $B_m$ but it is independent of $\epsilon$.

**Proof of Lemma 2.18.** For $m = 1, 2, \ldots, M$, write

$$U^m(x) := U^{\sigma_m}(\epsilon x + z_m), \forall x \in B_m.$$ 

Then we obtain

$$\left\{ \begin{array}{l}
(\Delta + \epsilon^2 \kappa^2)U^m(x) = \epsilon^2 (\Delta + \kappa^2)U^{\sigma_m}(\epsilon x + z_m) = 0, \ \text{for} \ x \in B_m, \\
U^m(\xi) = U^{\sigma_m}(\epsilon \xi + z_m) = \sigma(\epsilon \xi + z_m), \ \text{for} \ \xi \in \partial B_m,
\end{array} \right. \quad (2.119)$$

and also $\frac{\partial U^m}{\partial n_m(\xi)} = \nabla U^m(\xi) \cdot \nu_m(\xi) = \epsilon \nabla U^{\sigma_m}(\epsilon \xi + z_m) \cdot \nu_m(\epsilon \xi + z_m) = \epsilon \frac{\partial U^{\sigma_m}(\epsilon \xi + z_m)}{\partial n_m(\epsilon \xi + z_m)}$. Hence,

$$\left\| \frac{\partial U^m}{\partial n_m} \right\|_{L^2(\partial B_m)}^2 = \int_{\partial B_m} \left\| \frac{\partial U^m}{\partial n_m(\eta)} \right\|^2 \ d\eta$$

$$= \int_{\partial D_m} \epsilon^2 \left\| \frac{\partial U^{\sigma_m}(s)}{\partial n_m(s)} \right\|^2 \epsilon^{-2} ds, \ [s := \epsilon \eta + z_m]$$

$$= \left\| \frac{\partial U^{\sigma_m}}{\partial n_m} \right\|^2_{L^2(\partial D_m)},$$

which gives us

$$\left\| \frac{\partial U^{\sigma_m}}{\partial n_m} \right\|_{L^2(\partial D_m)} \leq C_7, \quad (2.120)$$

For every function $\zeta_m \in H^1(\partial D_m)$, the corresponding $U^{\zeta_m}$ exists on $D_m$ as mentioned in (2.115) and then the corresponding functions on $\bar{B}_m$ and the inequality (2.120) will be satisfied by all these functions. Let $A_{D_m} : H^1(\partial D_m) \to L^2(\partial D_m)$ and $A_{B_m} : H^1(\partial B_m) \to L^2(\partial B_m)$ be the Dirichlet to Neumann maps. Then we get the following estimate from (2.120).

$$\|A_{D_m}\|_{\mathcal{L}(H^1(\partial D_m), L^2(\partial D_m))} \leq \frac{1}{\epsilon} \|A_{B_m}\|_{\mathcal{L}(H^1(\partial B_m), L^2(\partial B_m))}.$$
This implies that,
\[
\left\| \frac{\partial U_{\sigma_m}^{\infty}}{\partial v_m} \right\|_{H^{-1}(\partial D_m)} \leq \left\| A_{D_m}^{\infty} \right\|_{L^2(\partial D_m)} \left\| L^2(\partial D_m), H^{-1}(\partial D_m) \right\| \\
= \left\| A_{D_m}^{\infty} \right\|_{L^2(\partial D_m)} \left\| L^2(\partial D_m) \right\| \\
\leq \frac{1}{\epsilon} \left\| A_{B_m} \right\|_{L^2(\partial B_m)} \left\| L^2(\partial B_m) \right\|.
\]
\[
(2.121)
\]
Now, by (2.113) and (2.115), we obtatin
\[
\left\| \frac{\partial U_{\sigma_m}^{\infty}}{\partial v_m} \right\|_{H^{-1}(\partial D_m)} \leq \hat{C}(\kappa) \left\| A_{B_m} \right\|_{L^2(\partial B_m)} \left\| L^2(\partial B_m) \right\|.
\]
\[
(2.122)
\]
Hence the result is true with \( C_7 := \hat{C}(\kappa) \left\| A_{B_m} \right\|_{L^2(\partial B_m)} \left\| L^2(\partial B_m) \right\| \) is bounded by a constant depending only on \( B_m \) through its size and Lipschitz character of \( B_m \), see Remark 2.23.

### 2.2.3. Further estimates on the total charge \( \int_{\partial D_m} \frac{\partial U_{\sigma_m}^{\infty}(s)}{\partial v_m(s)} ds, m = 1, \ldots, M \).

**Definition 2.19.** Similarly to Definition 2.7, we call \( \sigma_m \in L^2(\partial D_m) \) satisfying (2.73), the solution of the problem (1.1-1.3), as surface charge distributions. Using these surface charge distributions we define the total charge on each surface \( \partial D_m \) denoted by \( Q_m \) as

\[
Q_m := \int_{\partial D_m} \frac{\partial U_{\sigma_m}^{\infty}(s)}{\partial v_m(s)} ds.
\]
\[
(2.123)
\]
**Lemma 2.20.** For \( m = 1, 2, \ldots, M \), the absolute value of the total charge \( Q_m \) on each surface \( \partial D_m \) is bounded by \( \epsilon \), i.e.

\[
|Q_m| \leq \hat{\epsilon},
\]
\[
(2.124)
\]
where \( \hat{\epsilon} := |\partial B| \hat{C} \left\| A_{B_m} \right\|_{L^2(\partial B_m)} \) with \( \partial B \) and \( \hat{C} \) are defined in (2.104) and (2.111) respectively.

**Proof of Lemma 2.20.** From Proposition 2.14, we have the estimate for the surface charge distributions \( \sigma_m \) as \( \left\| \sigma_m \right\|_{L^2(\partial D_m)} \leq \hat{C}(\kappa) \epsilon \), with \( \hat{\epsilon}(\kappa) := \hat{C} |\partial B|^{-\frac{1}{2}} \), which results the Lemma 2.18 as

\[
\left\| \frac{\partial U_{\sigma_m}^{\infty}}{\partial v_m} \right\|_{H^{-1}(\partial D_m)} \leq \hat{C}(\kappa) \left\| A_{B_m} \right\|_{L^2(\partial B_m)} \left\| L^2(\partial B_m) \right\|.
\]

Hence

\[
|Q_m| = \left| \int_{\partial D_m} \frac{\partial U_{\sigma_m}^{\infty}(s)}{\partial v_m(s)} ds \right| \\
\leq \left| \int_{\partial D_m} \left\| \frac{\partial U_{\sigma_m}^{\infty}(s)}{\partial v_m(s)} \right\|_{H^{-1}(\partial D_m)} ds \right| \\
\leq \left| \int_{\partial D_m} \left\| \frac{\partial U_{\sigma_m}^{\infty}(s)}{\partial v_m(s)} \right\|_{L^2(\partial D_m)} ds \right| \\
\leq \epsilon |\partial B| \hat{\epsilon} \left\| A_{B_m} \right\|_{L^2(\partial B_m)} \left\| L^2(\partial B_m) \right\|.
\]

\[
(2.125)
\]

**Proposition 2.21.** The far-field pattern \( U^{\infty} \) corresponding to the scattered solution of the problem (1.1-1.3) has the following asymptotic expansion

\[
U^{\infty}(\tilde{x}) = \sum_{m=1}^{M} e^{-i\kappa\tilde{x} \cdot z_m} Q_m + O(\kappa a^2),
\]
\[
(2.125)
\]
with \( Q_m \) given by (2.123), if \( \kappa a < 1 \) where \( O(\kappa a^2) \leq \hat{C} \kappa a^2 \) and \( \hat{C} := \frac{|\partial B| \hat{C} \left\| A_{B_m} \right\|_{L^2(\partial B_m)}}{\max_{1 \leq m \leq M} \text{diam}(B_m)} \).
Proof of Proposition 2.21. From (2.117), we know that

\[ U^s(x) = \sum_{m=1}^{M} \int_{\partial D_m} \Phi(x, s) \frac{\partial U^{\sigma_m}(s)}{\partial \nu_m(s)} ds, \text{ for } x \in \mathbb{R}^3 \setminus \bigcup_{m=1}^{M} D_m. \]

Hence

\[ U^{\infty}(\hat{x}) = \sum_{m=1}^{M} \int_{\partial D_m} e^{-i\hat{k} \cdot \hat{s}} \frac{\partial U^{\sigma_m}(s)}{\partial \nu_m(s)} ds \]

\[ = \sum_{m=1}^{M} \left( e^{-i\hat{k} \cdot \hat{s}} \cdot z_m Q_m + \int_{\partial D_m} [e^{-i\hat{k} \cdot \hat{s}} - e^{-i\hat{k} \cdot \hat{s}} \cdot z_m] \frac{\partial U^{\sigma_m}(s)}{\partial \nu_m(s)} ds \right). \quad (2.126) \]

As in Lemma 2.20, we have from Lemma 2.18:

\[ \int_{\partial D_m} \frac{\partial U^{\sigma_m}(s)}{\partial \nu_m(s)} ds \leq \hat{C} a, \forall m = 1, 2, \ldots, M \quad (2.127) \]

with \( \hat{C} := \frac{\max_{1 \leq m \leq M} \text{diam}(B_m)}{\max_{1 \leq m \leq M} \text{diam}(B_m)} \). It gives us the following estimate in the similar lines of (2.57):

\[ \left| \int_{\partial D_m} [e^{-i\hat{k} \cdot \hat{s}} - e^{-i\hat{k} \cdot \hat{s}} \cdot z_m] \frac{\partial U^{\sigma_m}(s)}{\partial \nu_m(s)} ds \right| \leq \frac{1}{2} \hat{C} \kappa a^2 \frac{1}{2 - \frac{1}{2} \kappa a}, \text{ if } a < \frac{2}{\kappa \max} \left( \frac{2}{\kappa} \right) \quad (2.28) \]

which means

\[ \int_{\partial D_m} [e^{-i\hat{k} \cdot \hat{s}} - e^{-i\hat{k} \cdot \hat{s}} \cdot z_m] \frac{\partial U^{\sigma_m}(s)}{\partial \nu_m(s)} ds \leq \hat{C} \kappa a^2, \text{ for } a \leq \frac{1}{\kappa \max}. \quad (2.29) \]

Now substitution of (2.29) in (2.126) gives the required result (2.125).

D

Let us derive a formula for \( Q_m \). For \( s_m \in \partial D_m \), using the Dirichlet boundary condition (1.2), we have

\[ 0 = U^i(s_m) = U^i(s_m) + \sum_{j=1}^{M} \int_{\partial D_j} \Phi(s_m, s) \frac{\partial U^{\sigma_j}(s)}{\partial \nu_j(s)} ds \]

\[ = U^i(s_m) + \sum_{j=1}^{M} \left( \Phi(s_m, z_j) Q_j + \int_{\partial D_j} \Phi(s_m, s) - \Phi(s_m, z_j) \frac{\partial U^{\sigma_j}(s)}{\partial \nu_j(s)} ds \right) + \int_{\partial D_m} \Phi(s_m, s) \frac{\partial U^{\sigma_m}(s)}{\partial \nu_m(s)} ds. \quad (2.130) \]

Now let us estimate \( \int_{\partial D_j} [\Phi(s_m, s) - \Phi(s_m, z_j)] \frac{\partial U^{\sigma_j}(s)}{\partial \nu_j(s)} ds \) for \( j \neq m \).

For \( m, j = 1, \ldots, M, \) and \( j \neq m \), by making use of (2.60), (2.61) and (2.127) we obtain the below in the similar lines of (2.62):

\[ \left| \int_{\partial D_j} [\Phi(s_m, s) - \Phi(s_m, z_j)] \frac{\partial U^{\sigma_j}(s)}{\partial \nu_j(s)} ds \right| < \hat{C} \frac{a}{d} \left( \kappa + \frac{1}{d} \right) a. \quad (2.131) \]

Define, \( \Phi_0(s_m, s) := \frac{1}{4\pi |s_m - s|} \). Then (2.130) can be written as

\[ 0 = U^i(s_m) + \sum_{j=1}^{M} \Phi(s_m, z_j) Q_j + O \left( (M - 1) \left( \frac{\kappa a^2}{d} + \frac{a^2}{d^2} \right) \right) \]

\[ + \int_{\partial D_m} \Phi_0(s_m, s) \left[ 1 + (e^{i|s_m - s|} - 1) \right] \frac{\partial U^{\sigma_m}(s)}{\partial \nu_m(s)} ds. \quad (2.132) \]
By using the Taylor series expansions of the exponential term \(e^{in|m-s|}\), the above can also be written as,

\[
\int_{\partial D_m} \Phi_0(s_m, s) \frac{\partial U^{\sigma_m}(s)}{\partial \nu_m(s)} ds + O(\kappa a) = -U^i(s_m) - \sum_{j=1}^{M} \Phi(s_m, z_j)Q_j + O\left((M-1)\left(\frac{\kappa a^2}{d} + \frac{a^2}{d^2}\right)\right).
\]

(2.133)

Indeed, for \(m = 1, \ldots, M\), we have the below estimate in the similar lines of (2.65):

\[
\left| \int_{\partial D_m} \Phi_0(s_m, s) \left(e^{in|m-s|} - 1\right) \frac{\partial U^{\sigma_m}(s)}{\partial \nu_m(s)} ds \right| \leq \tilde{C}\kappa a, \text{ for } a \leq \frac{1}{\kappa_{\text{max}}}.
\]

(2.134)

Define \(U_m := \int_{\partial D_m} \Phi_0(s_m, s) \frac{\partial U^{\sigma_m}(s)}{\partial \nu_m(s)} ds\), \(s_m \in \partial D_m\). Then (2.133) can be written as

\[
U_m = -U^i(s_m) - \sum_{j=1}^{M} \Phi(s_m, z_j)Q_j + O(\kappa a) + O\left((M-1)\left(\frac{\kappa a^2}{d} + \frac{a^2}{d^2}\right)\right).
\]

(2.135)

For \(m = 1, \ldots, M\), let \(\tilde{\sigma}_m \in L^2(\partial D_m)\) be the surface charge distributions which define,

- The constant potentials \(\tilde{U}_m\) as

\[
\int_{\partial D_m} \Phi_0(s_m, s) \frac{\partial U^{\sigma_m}(s)}{\partial \nu_m(s)} ds = \tilde{U}_m := -U^i(z_m) - \sum_{j=1}^{M} \Phi(z_m, z_j)Q_j.
\]

(2.136)

- The total charge on the surface \(\partial D_m\) as

\[
\tilde{Q}_m := \int_{\partial D_m} \frac{\partial U^{\sigma_m}(s)}{\partial \nu_m(s)} ds.
\]

We set the electrical capacitance \(C_m\) for \(1 \leq m \leq M\) as

\[
C_m = \frac{\tilde{Q}_m}{\tilde{U}_m}.
\]

Following in the similar lines of the proofs of Lemma 2.10, Lemma 2.11 and Proposition 2.12 concerning the single layer potentials, we can prove the following results in the case of double layer potentials.

- The consecutive pairwise difference between \(\frac{\partial U^{\sigma_m}}{\partial \nu_m}, \frac{\partial U^{\sigma_m}}{\partial \nu_m}, \tilde{Q}_m, \tilde{Q}_m\) have the following behaviour.

\[
\left| \frac{\partial U^{\sigma_m}}{\partial \nu_m} - \frac{\partial U^{\sigma_m}}{\partial \nu_m} \right|_{H^{-1}(\partial D_m)} = O\left(\kappa a + (M-1)\left(\frac{\kappa a^2}{d} + \frac{a^3}{d^2}\right)\right),
\]

(2.137)

\[
Q_m - \tilde{Q}_m = O\left(\kappa a^2 + (M-1)\left(\frac{\kappa a^3}{d} + \frac{a^3}{d^2}\right)\right).
\]

(2.138)

- For every \(1 \leq m \leq M\), we have

\[
\bar{C}_m = \frac{\tilde{C}_B_m}{\text{diam}(B_m)} a \text{ and } \tilde{Q}_m = \frac{\tilde{Q}_B_m}{\text{diam}(B_m)} a.
\]

- For \(m = 1, 2, \ldots, M\), the total charge \(\tilde{Q}_m\) on each surface \(\partial D_m\) of the small scatterer \(D_m\) can be calculated from the algebraic system

\[
\frac{\tilde{Q}_m}{C_m} := -U^i(z_m) - \sum_{j=1}^{M} C_j \Phi(z_m, z_j) \frac{\tilde{Q}_j}{C_j} + O\left((M-1)\frac{\kappa a^2}{d} + (M-1)^2\left(\frac{\kappa a^3}{d} + \frac{a^3}{d^2}\right)\right).
\]

(2.139)
2.3. The algebraic system. Define the algebraic system,

\[ \frac{\tilde{Q}_m}{C_m} := -U^i(z_m) + \sum_{j=1}^{M} \tilde{C}_j \Phi(z_m, z_j) \frac{\tilde{Q}_j}{C_j}, \]

for all \( m = 1, 2, \ldots, M \). It can be written in a compact form as

\[ B \tilde{Q} = U^I, \]

where \( \tilde{Q}, U^I \in \mathbb{C}^{M \times 1} \) and \( B \in \mathbb{C}^{M \times M} \) are defined as

\[
B := \begin{pmatrix}
-\frac{1}{C_1} & -\Phi(z_1, z_2) & -\Phi(z_1, z_3) & \cdots & -\Phi(z_1, z_M) \\
-\Phi(z_2, z_1) & -\frac{1}{C_2} & -\Phi(z_2, z_3) & \cdots & -\Phi(z_2, z_M) \\
& \ddots & \ddots & \ddots & \ddots \\
-\Phi(z_M, z_1) & -\Phi(z_M, z_2) & \cdots & -\Phi(z_M, z_{M-1}) & -\frac{1}{C_M}
\end{pmatrix},
\]

\[
\tilde{Q} := (\tilde{Q}_1, \tilde{Q}_2, \ldots, \tilde{Q}_M) \top \quad \text{and} \quad U^I := (U^i(z_1), U^i(z_2), \ldots, U^i(z_M)) \top.
\]

The above linear algebraic system is solvable for \( \tilde{Q}_j, 1 \leq j \leq M \), when the matrix \( B \) is invertible. Next we discuss the possibilities of this invertibility,

- If

\[
\max_{1 \leq m \leq M} \sum_{j \neq m} \frac{\tilde{C}_m}{|z_m - z_j|} < 4\pi,
\]

then \( B \) satisfies the diagonally dominant condition and hence the system is solvable. Since, \( \tilde{C}_m := \epsilon \tilde{C}_B_m = \max\limits_{1 \leq m \leq M} \frac{C_{B_m}}{diam(B_m)} a \) then (2.142) is valid if \( (M - 1) \frac{a}{\pi} < 4\pi \left( \max_{1 \leq m \leq M} \tilde{C}_B_m \right)^{-1} \max_{1 \leq m \leq M} diam(B_m) \), where \( \tilde{C}_B_m \) depends only the Lipschitz character of \( B_m \).

- If \( a < cd \) for some constant \( c \), i.e. precisely if \( \max_{1 \leq m \leq M} \tilde{C}_m < \frac{\epsilon d}{3\pi} \) and \( t := \min_{j \neq m, 1 \leq j, m \leq M} \cos(|z_m - z_j|) \geq 0 \), then the matrix \( B \) is invertible. Remark that \( \epsilon c \leq \frac{\epsilon d}{3\pi} \left( \max_{1 \leq m \leq M} \tilde{C}_B_m \right)^{-1} \max_{1 \leq m \leq M} diam(B_m) \). We state this invertibility property in the following lemma.

**Lemma 2.22.** If \( \max_{1 \leq m \leq M} \tilde{C}_m < \frac{\epsilon d}{3\pi} \) and \( t := \min_{j \neq m, 1 \leq j, m \leq M} \cos(|z_m - z_j|) \geq 0 \), then the matrix \( B \) is invertible and the solution vector \( \tilde{Q} \) of (2.141) satisfies the estimate

\[
\sum_{m=1}^{M} |\tilde{Q}_m|^2 \tilde{C}_m^{-1} \leq 4 \left( 1 - \frac{3t}{5\pi d} \max M \max_{m=1} \tilde{C}_m \right)^{-2} \sum_{m=1}^{M} |U^i(z_m)|^2 \tilde{C}_m.
\]

**Proof of Lemma 2.22.** The idea of the proof of this lemma is given by Maz'ya and Movchan in [16] for the case where \( \Phi \) is the Green’s function of the Dirichlet-Laplacian in a bounded domain. We adapt their argument to the case of Helmholtz, \( \kappa \neq 0 \), on the whole space. For the reader’s convenience, we give the detailed proof in the appendix.

2.4. End of the proof of Theorem 1.2. We can rewrite the equation (2.143) using norm inequalities as

\[
\sum_{m=1}^{M} |\tilde{Q}_m| \leq 2 \left( 1 - \frac{3t}{5\pi d} \max M \max_{m=1} \tilde{C}_m \right)^{-1} M \max_{m=1} |\tilde{Q}_m| \max_{m=1} |U^i(z_m)|.
\]
with \( t := \min_{j \neq m, 1 \leq j, m \leq M} \cos(\kappa |z_m - z_j|) \). The difference between (2.72)/(2.139) and (2.140) produce the following

\[
\frac{\hat{Q}_m - \check{Q}_m}{C_m} = -\sum_{j=1}^{M} \Phi(z_m, z_j) \left( \hat{Q}_j - \check{Q}_j \right) + O \left( (M-1) \frac{\kappa a^2}{d} + (M-1)^2 \left( \frac{\kappa a^3}{d^2} + \frac{a^3}{d^3} \right) \right),
\]

(2.145)

for \( m = 1, \ldots, M \). Comparing the above system of equations (2.145) with (2.140) and by making use of the estimate (2.144), we obtain

\[
\sum_{m=1}^{M} (\hat{Q}_m - \check{Q}_m) = O \left( M(M-1) \frac{\kappa a^3}{d} + M(M-1)^2 \left( \frac{\kappa a^4}{d^2} + \frac{a^4}{d^3} \right) \right).
\]

(2.146)

We can evaluate the \( \check{Q}_m \)'s from the algebraic system (2.140). Hence, by using (2.69)/(2.138) and (2.146) in (2.53)/(2.125) we can represent the far-field pattern in terms of \( \check{Q}_m \) as below:

\[
U^\infty(\mathbf{x}) = \sum_{m=1}^{M} [e^{-in\hat{x} \cdot z_m} Q_m + O(\kappa a^2)]
\]

\[
= \sum_{m=1}^{M} [e^{-in\hat{x} \cdot z_m} (\hat{Q}_m + (Q_m - \hat{Q}_m) + (Q_m - \check{Q}_m)] + O(\kappa a^2)]
\]

\[
= \sum_{m=1}^{M} e^{-in\hat{x} \cdot z_m} \hat{Q}_m + O \left( M \kappa a^2 + M(M-1) \left( \frac{\kappa a^3}{d} + \frac{a^3}{d^2} \right) + M(M-1)^2 \frac{a^4}{d} \left( \frac{\kappa a^4}{d^2} + \frac{a^4}{d^3} \right) \right).
\]

(2.147)

Hence Theorem 1.2 is proved by setting \( \hat{\sigma}_m := \frac{\sigma_m}{U_m} \) as the surface density which defines \( \check{Q}_m \). Finally, let us stress that

1. The constant \( \hat{c} = \left[ \frac{\left( \frac{c+1}{4} \right) |\partial B|}{\kappa_m \max_{1 \leq m \leq M} \gamma_m} \right]^{-\frac{1}{2}} \) appearing in Proposition 2.14 will serve our purpose in Theorem 1.2 by defining \( c_0 := \hat{c} \max_{1 \leq m \leq M} \text{diam}(B_m) \) when we use double layer potentials. Also observe that the constant \( c \) appearing in Proposition 2.2 can be used to prove Theorem 1.2 if we replace the second condition in (1.8) by the stronger one \( (M-1)^2 \frac{a^4}{d} \leq c_0 \) with \( c_0 := c \max \text{diam}(B_m) \).

2. The coefficients \( \hat{\sigma}_m, \check{Q}_m, \check{C}_m \) play the roles of \( \sigma_m, Q_m, C_m \) respectively in Theorem 1.2.

3. The constant appearing in \( O \left( M \kappa a^2 + M(M-1) \left( \frac{\kappa a^3}{d} + \frac{a^3}{d^2} \right) + M(M-1)^2 \frac{a^4}{d} \left( \frac{\kappa a^4}{d^2} + \frac{a^4}{d^3} \right) \right) \) is \( (\hat{C} + \max_{1 \leq m \leq M} \gamma_m) \), where \( \hat{C} \) and \( C_m \) are defined in Lemma 2.6 and Proposition 2.21.

4. The constant \( a_0 \) appearing in (1.8) of Theorem 1.2 is the minimum among

\[
\frac{1}{\kappa_{\text{max}}} \min \left\{ 1, \sqrt[3]{\frac{4\pi}{3}} \right\}, \quad \frac{2\pi}{\kappa_{\text{max}}} \max_{1 \leq m \leq M} \text{diam}(B_m) \]

and

\[
\frac{2\sqrt{\pi}}{\kappa_{\text{max}}} \max_{1 \leq m \leq M} \text{diam}(B_m) \]

\[
\kappa_{\text{max}} \left( \partial B \right) \max_{1 \leq m \leq M} \left\| \left( \frac{1}{4\pi t + \gamma_m^{(m)}} \right)^{-1} \right\|_{L^2(\partial B_m), L^2(\partial B_m)} \]

5. The constant \( c_1 \) appearing in (1.12) of Theorem 1.2 is \( \frac{2\pi}{\kappa_{\text{max}}} \max_{1 \leq m \leq M} \text{diam}(B_m) \) and it follows from Lemma 2.11 and Lemma 2.22.

From the last points, we see that the constants appearing in Theorem 1.2 depend only on \( d_{\text{max}}, \kappa_{\text{max}} \) and \( B_m \)'s through their diameters, capacitances and the norms of the boundary operators \( S_{B_m}^{-1} : H^1(\partial B_m) \rightarrow H^1(\partial B_m) \).
let \( L^2(\partial B_m) \), \((\frac{1}{2}I + D^2_B)^{-1} : L^2(\partial B_m) \rightarrow L^2(\partial B_m) \) and \( A_{B_m} : H^1(\partial B_m) \rightarrow L^2(\partial B_m) \). In the following remark, we show how the dependency on \( B_m \)'s are actually only through their Lipschitz character.

**Remark 2.23.**

1. Let \( B \) be a bounded, simply connected and Lipschitz domain in \( \mathbb{R}^3 \). The quantities \( \left\| \left( \frac{1}{2}I + D^2_B \right)^{-1} \right\|_{L(2(\partial B),L^2(\partial B))} \) and \( \|A_B\|_{L(\mathcal{H}^1(\partial B),L^2(\partial B))} \) depend only on the Lipschitz character of \( B \). Indeed, we first remark that \( A_B = \left( \frac{1}{2}I + D^2_B \right) (S^{\alpha})^{-1} \) hence \( \|A_B\|_{L(\mathcal{H}^1(\partial B),L^2(\partial B))} \leq \left\| \left( \frac{1}{2}I + D^2_B \right)^{-1} \right\|_{L(\mathcal{H}^1(\partial B),L^2(\partial B))} \|S^{\alpha}\|_{L(\mathcal{H}^1(\partial B),L^2(\partial B))} \). The operators \( \frac{1}{2}I + D^2_B \) and \( S^{\alpha} \) are isomorphism in the mentioned spaces and their norms depend only on the Lipschitz character of \( B \), see Section 1 in [4] for instance.

2. The capacitance \( C_B \) of a bounded, connected and Lipschitz domain \( B \) of \( \mathbb{R}^3 \) depends only on the Lipschitz character of \( B \). Indeed, since \( C_B := \int_{\partial B} \sigma(s)ds \) where \( \int_{\partial B} \frac{\sigma(t)}{r-t}dt = 1, \ s \in \partial B, \) then from the invertibility of the single layer potential \( S^{\alpha} : L^2(\partial B) \rightarrow H^1(\partial B) \), we deduce that

\[
C_B \leq |\partial B|^\frac{1}{2} \|\sigma\|_{L^2(\partial B)} \leq |\partial B|^\frac{1}{2} \|{(S^{\alpha})^{-1}}\|_{L(\mathcal{H}^1(\partial B),L^2(\partial B))}\|\partial B\|^\frac{1}{2} = \|(S^{\alpha})^{-1}\|_{L(\mathcal{H}^1(\partial B),L^2(\partial B))}\|\partial B\|
\]

On the other hand, we recall the following lower estimate, see Theorem 3.1 in [20] for instance,

\[
C_B \geq \frac{4\pi|\partial B|^2}{J}
\]

where \( J := \int_{\partial B} \int_{\partial B} \frac{1}{|s-t|}dsdt \). Remark that \( J = 4\pi \int_{\partial B} S^{\alpha}(1)(s)ds \). Hence

\[
J \leq 4\pi|\partial B|^\frac{1}{2} \|{(S^{\alpha})^{-1}}\|_{L(\mathcal{H}^1(\partial B),L^2(\partial B))}\|\partial B\| \leq 4\pi \|{(S^{\alpha})^{-1}}\|_{L(\mathcal{H}^1(\partial B),L^2(\partial B))}\|\partial B\|
\]

and using (2.149) we obtain the lower bound

\[
C_B \geq \|{(S^{\alpha})^{-1}}\|_{L(\mathcal{H}^1(\partial B),L^2(\partial B))}\|\partial B\|.
\]

Finally combining (2.148) and (2.150), we derive the estimated

\[
\|{(S^{\alpha})^{-1}}\|_{L(\mathcal{H}^1(\partial B),L^2(\partial B))}\|\partial B\| \leq C_B \leq \|{(S^{\alpha})^{-1}}\|_{L(\mathcal{H}^1(\partial B),L^2(\partial B))}\|\partial B\|
\]

which shows, in particular, that the capacitance of \( B \) depends only on the Lipschitz character \( B \), knowing that also \( |\partial B| \) can be estimated only with the Lipschitz character of \( B \), see for instance the observations in Remark 2.5 in [1].

**2.5. Proof of Remark 1.4.** For \( m = 1, \ldots , M \), fixed, we distinguish between the obstacles \( D_j, j \neq m \) which are near to \( D_m \) from the ones which are far from \( D_m \) as follows. Let \( \Omega_m, 1 \leq m \leq M \) be the balls of center \( z_m \) and of radius \( (\frac{1}{\lambda} + d^m) \) with \( 0 < \alpha \leq 1 \). The bodies lying in \( \Omega_m \) will fall into the category, \( N_m \), of near by obstacles and the others into the category, \( F_m \), of far obstacles to \( D_m \). Since the obstacles \( D_m \) are balls with same diameter, the number of obstacles near by \( D_m \) will not exceed \( \left( \frac{\alpha + 2\alpha}{\alpha + d} \right)^3 \left[ \frac{\pi}{\frac{4\pi}{2}(\alpha + d)/2} \right]^3 \).

With this observation, instead of (1.9)/(2.147), the far field will have the asymptotic expansion (1.16). Indeed,

- For the bodies \( D_j \in N_m, j \neq m \) we have the estimate (2.62) but for the bodies \( D_j \in F_m \), we obtain the following estimate

\[
\int_{\partial D_j} \left[ \Phi(s_m, s) - \Phi(s_m, z_j) \right] \sigma_j(s)ds \leq C_n \left( \frac{\alpha}{\alpha + d} \right) a.
\]

- Due to the estimates (2.62) and (2.152), while representing the scattered field in terms of single layer potential, corresponding changes will take place in (2.63), (2.64), (2.66), (2.69) and in (2.72) which turn modify the asymptotic expansion (1.9)/(2.147) as follows

\[
U^\infty(x, \theta) = \sum_{m=1}^{M} e^{-in\hat{z} \cdot z_m} Q_m + O \left( M \frac{\ka^2}{d^2} \frac{M(M-1)}{} \left( \frac{\ka^3}{d^2} + \frac{\ka^3}{d^2} \right) M \left( \frac{\ka^2}{d^2} + \frac{\ka^3}{d^2} \right) \right)
\]
Let us consider that all the small bodies are included in a large bounded domain Ω. We distinguish the

generators

Given the far-field pattern

Inverse Problem: for a given incident direction

with the error of order

\( O(t_m) \) double inclusions (1.18) and the fact that

d \( t_m \)'s are uniformly bounded from below by a positive constant.

\[ \frac{a + 2d^\alpha}{a + d} \]

which can be used to derive (1.16) from (2.153).

Finally, it is easily seen that the above analysis applies also for non-flat Lipschitz domains \( D_m \) by using the
double inclusions (1.18) and the fact that \( t_m \)'s are uniformly bounded from below by a positive constant.

\[ \frac{a + 2d^\alpha}{a + d} \]

3. The inverse problem. From (2.147), we can write the far-field pattern as

\[ U^\infty(\hat{x}) = \sum_{m=1}^{M} e^{-iK^\infty z_m} \tilde{Q}_m \]

with the error of order

\[ O\left(M^2 \alpha^2 + (M - 1)^2 \frac{a^3}{d^2} + M(M - 1)^2 \frac{a^3}{d^2} \right) \]

and \( \tilde{Q}_m \) can be obtained from the Foldy type system (2.141). Let us denote the inverse of \( B \) by \( B \), then we can state the

following expression of the far field pattern;

\[ U^\infty(\hat{x}, \theta) = \sum_{m=1}^{M} \sum_{j=1}^{M} B_{m,j} e^{-iK^\infty z_m} e^{i\theta z_j} + O\left(M^2 \alpha^2 + (M - 1)^2 \frac{a^3}{d^2} + M(M - 1)^2 \frac{a^3}{d^2} \right) \]

for a given incident direction \( \theta \) and observation direction \( \hat{x} \).

Our main focus is the following inverse problem.

Inverse Problem: Given the far-field pattern \( U^\infty(\hat{x}, \theta) \) for several incident and observation directions

\( \theta \) and \( \hat{x} \), find the locations \( z_1, z_2, \ldots, z_M \) and reconstruct the capacitances \( C_1, C_2, \ldots, C_M \) of the small

scatterers \( D_1, D_2, \ldots, D_M \). respectively.

Let us consider that all the small bodies are included in a large bounded domain \( \Omega \). We distinguish the

following two cases in terms of the minimum distance \( d := d(\epsilon) \):

- Case 1: We have \( d \ll 1 \) and \( (M - 1) \frac{a}{d} \ll 1 \). In this case, using the formula (3.2) we deduce that

\[ U^\infty(\hat{x}, \theta) = \sum_{m=1}^{M} \sum_{j=1}^{M} B_{m,j} e^{-iK^\infty z_m} e^{i\theta z_j} + o\left((M - 1) \frac{a^2}{d^2}\right) \]

as \( \epsilon \to 0 \). Indeed, since \( \tilde{Q}_m = \sum_{j=1}^{M} B_{m,j} e^{i\theta z_j} \), then

\[ U^\infty(\hat{x}, \theta) = \sum_{m=1}^{M} e^{-iK^\infty z_m} \tilde{Q}_m + O\left(M^2 \alpha^2 + (M - 1)^2 \frac{a^3}{d^2} + M(M - 1)^2 \frac{a^3}{d^2} \right) \]

\[ = \sum_{m=1}^{M} \sum_{j=1}^{M} B_{m,j} e^{i\theta z_j} + O\left(M^2 \alpha^2 + (M - 1)^2 \frac{a^3}{d^2} \right) \]

\[ = \sum_{m=1}^{M} \sum_{j=1}^{M} B_{m,j} e^{i\theta z_j} + o\left((M - 1) \frac{a^2}{d} \right). \]
Remark that \( \tilde{Q}_m := -C_m U^i(z_m) + C_m \sum_{j=1}^{M} \Phi(z_m, z_j) \hat{C}_j U^i(z_j) + O \left( (M-1)^2 \frac{a^3}{\epsilon^3} \right) \), where \( C_m U^i(z_m) \) behaves as \( a \) and \( \tilde{C}_m \sum_{j \neq m}^{M} \Phi(z_m, z_j) \hat{C}_j U^i(z_j) \) behaves as \( (M-1) \frac{a^2}{\epsilon^2} \). This means that (3.3) reduces to

\[
U^\infty(\hat{x}, \theta) = - \sum_{m=1}^{M} C_m e^{-i k \hat{x} \cdot z_m} e^{i k \epsilon \theta \cdot z_m} + \sum_{m=1}^{M} \sum_{j \neq m}^{M} \tilde{C}_m \hat{C}_j \Phi(z_m, z_j) e^{-i k \hat{x} \cdot z_m} e^{i k \epsilon \theta \cdot z_j} + o \left( (M-1) \frac{a^2}{d} \right),
\]

where the first term models the Born approximation and second term models the first order interaction between the scatterers. As a conclusion, when we use (3.3) we compute the field generated by the first interaction between the collection of the scatterers \( z_m, m = 1, \ldots, M \).

- Case 2: We assume that there exists a positive constant \( d_0 \) such that \( d_0 \leq d \). In this case, using the formula (3.2) we have

\[
U^\infty(\hat{x}, \theta) = - \sum_{m=1}^{M} C_m e^{-i k \hat{x} \cdot z_m} e^{i k \epsilon \theta \cdot z_m} + O \left( \kappa a^2 \right)
\]

as \( \epsilon \to 0 \). In this case, we have only the Born approximation.

Based on the formulas (3.3) and (3.4), we setup a MUSIC type algorithm to locate the points \( z_j, j = 1, \ldots, M \) and then estimate the sizes of the scatterers \( D_j \). An example of distribution of the scatterers in case 1 is \( M = e^{-\frac{\kappa}{d}} \) and \( d(\epsilon) = \epsilon^7 \). Note that in case 2 we have a lower bound on the distances between the scatterers. This explains why we are in the Born regime. Remark also that, in this case, \( M \) is uniformly bounded since the obstacles are included in the bounded domain \( \Omega \).

### 3.1. Localisation of \( D_m \)'s via the MUSIC algorithm

The MUSIC algorithm is a method to determine the locations \( z_m, m = 1, 2, \ldots, M \), of the scatterers \( D_m, m = 1, 2, \ldots, M \) from the measured far-field pattern \( U^\infty(\hat{x}, \theta) \) for a finite set of incidence and observation directions, i.e. \( \hat{x}, \theta \in \{ \theta_j, j = 1, \ldots, N \} \subset \mathbb{S}^2 \). We refer the reader to the monographs [5] and [15] for more information about this algorithm. We follow the way presented in [15]. We assume that the number of scatterers is not larger than the number of incident and observation directions, i.e. \( N \geq M \). We define the response matrix \( F \in \mathbb{C}^{N \times N} \) by

\[
F_{jl} := U^\infty(\theta_j, \theta_l).
\]

From (3.2) and (3.5), we can write

\[
F_{jl} = \sum_{m=1}^{M} \sum_{j=1}^{M} \mathcal{B}_{mj} e^{-i k \theta_j \cdot z_m} e^{i k \theta_l \cdot z_j}
\]

\[
= \left[ e^{-i k \theta_j \cdot z_1}, e^{-i k \theta_j \cdot z_2}, \ldots, e^{-i k \theta_j \cdot z_M} \right] \cdot \mathcal{B} \cdot \left[ e^{i k \theta_l \cdot z_1}, e^{i k \theta_l \cdot z_2}, \ldots, e^{i k \theta_l \cdot z_M} \right]^{T}
\]

(3.6)

for all \( j, l = 1, \ldots, N \). We can factorize the response matrix \( F \) as below,

\[
F = H^* \mathcal{B} H
\]

(3.7)

where \( H \) is a complex matrix of order \( M \times N \) given by

\[
H_{jl} := e^{i k \theta_l \cdot z_j}
\]

for all \( j, l = 1, \ldots, N \). In order to determine the locations \( z_m \), we consider a 3d-grid of sampling points \( z \in \mathbb{R}^3 \) in a region containing the scatterers \( D_1, D_2, \ldots, D_M \). For each point \( z \), we define the vector \( \phi_z \in \mathbb{C}^N \) by

\[
\phi_z := (e^{-i k \theta_1 \cdot z}, e^{-i k \theta_2 \cdot z}, \ldots, e^{-i k \theta_N \cdot z})^{T},
\]

(3.8)

\(^5\)Since \( \kappa \leq \kappa_{\max}, \frac{d_0}{d} = \frac{\kappa}{d} \ll 1, \) and \( (M-1)^{\frac{2}{d}} \ll 1. \)

\(^6\)Since \( (M-1)^{\frac{2}{d}} \ll 1, M \alpha^2 = (M-1)^{\frac{2}{d}} \frac{M^2}{M^2-1} d \) and \( d \ll 1. \)
3.1.1. MUSIC characterisation of the response matrix. Recall that MUSIC is essentially based on characterizing the range of the response matrix $F$ (signal space), forming projections onto its null (noise) spaces, and computing its singular value decomposition. In other words, the MUSIC algorithm is based on the property that $\phi_z$ is in the range $\mathcal{R}(F)$ of $F$ if and only if $z$ is at one of locations of the scatterers. Precisely, let $P$ be the projection onto the null space $\mathcal{N}(F^*) = \mathcal{R}(F)^\perp$ of the adjoint matrix $F^*$ of $F$, then

$$z \in \{z_1, z_2, \ldots, z_M\} \iff P\phi_z = 0.$$  

It can be proved based on the non-singularity of the scattering matrix $B$ in the factorization (3.7) of $F$. Due to this, the standard linear algebraic argument yields that, if $N \geq M$ and the if the matrix $H$ has maximal rank $M$, then the ranges $\mathcal{R}(H^*)$ and $\mathcal{R}(F)$ coincide.

For sufficiently large number $N$ of incident and the observational directions by following the same lines as in [6,8,14,15], the maximal rank property of $H$ can be justified. In this case MUSIC algorithm is applicable for our response matrix $F$.

From the above discussion, MUSIC characterization of the locations of the small scatterers in acoustic exterior Dirichlet problem can be written as the following and is valid if the number of the observational and incidental directions is sufficiently large.

Theorem 3.1. Suppose $N \geq M$; then

$$z \in \{z_1, \ldots, z_M\} \iff \phi_z^j \in \mathcal{R}(H^*) = \mathcal{R}(F) \iff P\phi_z = 0,$$

where $P : \mathbb{C}^N \rightarrow \mathcal{R}(F)^\perp = \mathcal{N}(F^*)$ is the orthogonal projection onto the null space $\mathcal{N}(F^*)$ of $F^*$.

Let us point out that the orthogonal projection $P\phi_z$ in the MUSIC does not contain any information about the shape and the orientation of the small scatterers. Yet, if the locations $z_m$ of the scatterers $D_m$ are found, approximately, via the observation of the pseudo norms of $P\phi_z$, using the factorization (3.7) of the response matrix $F$, then one can retrieve the capacitances $C_m$ for $m = 1, \ldots, M$.

3.2. Recovering the capacitances and estimating the sizes of the scatterers. Once we locate the scatterers from the given far-field patterns using the MUSIC algorithm, we can recover the capacitances $C_m$ of $D_m$ from the factorization (3.7) of $F \in \mathbb{C}^{N \times N}$ as hinted earlier in Section 3.1.1.

Indeed, we know that the matrix $H$ has maximal rank, see Theorem 4.1 of [15] for instance. So, the matrix $HH^* \in \mathbb{C}^{M \times M}$ is invertible. Let us denote its inverse by $I_H$. Once we locate the scatterers through finding the locations $z_1, z_2, \ldots, z_M$ by using the MUSIC algorithm for the given far-field patterns, we can recover $I_H$ and hence the matrix $B \in \mathbb{C}^{M \times M}$ given by $B = I_H H F H^* I_H$, where $I_H H$ is the pseudo inverse of $H^*$. As we know the structure of $B \in \mathbb{C}^{M \times M}$, the inverse of $B \in \mathbb{C}^{M \times M}$, we can recover the capacitances $C_1, C_2, \ldots, C_M$ of the small scatterers $D_1, D_2, \ldots, D_M$ from the diagonal entries of $B$. From these capacitances, we can estimate the size of the obstacles. Indeed, assume that $D_j$’s are discs of radius $\rho_j$, and center 0 for simplicity, then we know that $\int_{\{y:|y|=\rho_j\}} \frac{ds}{|x-y|} = 4\pi \rho_j$, for $|x| = \rho$, as observed in [16, formula (5.12)]. Hence $\sigma_j(s) = \rho_j^{-1}$ and then $C_j = \int_{D_j} \rho_j^{-1} ds = 2\pi \rho_j$ from which we can estimate the radius $\rho_j$. Other geometries, as cylinders, for which one can estimate exactly the size are shown in chapter 4 of [20].

For general geometries, we can use the estimate (2.151) to provide lower and upper estimates of the size of scatterers. For this, one should estimate the norms of the operators appearing in (2.151) in terms of (only) the size of $\partial D_j$ and the Lipschitz smoothness character $L_j$ defined in the beginning of the introduction. This issue will be considered in a future work.

3.3. Numerical results and discussions. In order to illustrate relevant features of the method reviewed in the previous section, several computations have been performed, and the typical results acquired and presented here in. To generate the far-field data we numerically solve the exterior acoustic Dirichlet problem (1.1-1.3) via a Galerkin method, see [12,13]. For our calculations, we considered 50 incident and the observational directions obtained from the Gauss-Legendre polynomial. Let $d_{GL}$ stands for the degree of Gauss-Legendre polynomial. Then these $2d_{GL}^2(= 50)$ incident and the observational directions are obtained from the Gauss-Legendre polynomial of degree $d_{GL}(= 5)$, i.e. if we denote the zeros of the Gauss-Legendre polynomial of degree by $GL_k$, for $k = 1, \ldots, d_{GL}$ then the azimuth and the zenith angles $\theta$ and $\phi$ are given by

$$\phi = \cos^{-1}(GL_k), \quad k = 1, \ldots, d_{GL}.$$
\[
\theta = j \cdot \frac{\pi}{d_{GL}}, \quad j = 0, 1, \ldots, 2d_{GL} - 1.
\]

Combinations of these spherical coordinates will allow us to find the incident and the observational directions given by \((\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi)\). These directions are shown in Fig:3.1.

![Fig. 3.1. Incident and the observational directions.](image)

Due to the fact that MUSIC allows us to find the location of scatterers but not the size and shape, we presented the numerical results that are related to the balls of various size with various centers. Of course, we are able to locate the different type of scatterers as well. In Fig:3.2 we have shown the results with 10% random noise in the measured far-field performed on the balls with radius 0.5, and having the centers at \(A=(0,0,0)\), \(B=(1.5,1.5,1.5)\), \(C=(1.5,1.5,-1.5)\), \(D=(-1.5,-1.5,1.5)\) and at \(E=(-1.5,-1.5,-1.5)\). Fig:3.2(a) and Fig:3.2(d) are results performed for the single small scatterer having center at origin A. Here the first one shows the distribution of the singular values and second one, iso-surface of the pseudonormal values, shows the corresponding location of the scatterer. Fig:3.2(b) and Fig:3.2(c) are the distributions of the singular values while considering the two and five small scatterers namely \(A,D\) and \(A,B,C,D,E\) respectively. Fig:3.2(e) and Fig:3.2(f) are the corresponding isosurface plots to locate the scatterers.

From these figures we got the good reconstruction to locate the scatterers in a nice way. In general we can observe that, within our assumptions, MUSIC algorithm allows us to the locate the scatterers in a more finer way in presence of less noise while as the noise increases due to the noise location of the scatterer will get disturbed.

To finish this section, let us mention that the reconstruction depends on the choice of the signal and noise subspaces of the multi scale response matrix. For small measurement noise (or) for higher SNR it is easy to choose these subspaces otherwise it become hard due to the smooth distribution of the singular values.

4. Appendix: Proof of Lemma 2.22. We start by factorizing \(B\) as \(B = -(I + B_n C)C^{-1}\) where \(C := Diag(C_1, C_2, \ldots, C_M) \in \mathbb{C}^{M \times M}\), \(I\) is the identity matrix and \(B_n := -C^{-1}B\). Hence, the solvability of the system (2.141), depends on the existence of the inverse of \((I + B_n C)\). We have \((I + B_n C) : \mathbb{C}^M \to \mathbb{C}^M\), so it is enough to prove the injectivity in order to prove its invertibility. For this purpose, let \(X, Y\) are vectors in \(\mathbb{C}^M\) and consider the system

\[
(I + B_n C)X = Y.
\]

Let \((\cdot)^{real}\) and \((\cdot)^{img}\) denotes the real and the imaginary parts of the corresponding complex number/vector/matrix. Now, the following can be written from (4.1);

\[
(I + B_n^{real} C)X^{real} - B_n^{img} C X^{img} = Y^{real},
\]

\[
(I + B_n^{real} C)X^{img} + B_n^{img} C X^{real} = Y^{img},
\]

which leads to

\[
\langle (I + B_n^{real} C)X^{real}, CX^{real} \rangle - \langle B_n^{img} C X^{img}, CX^{real} \rangle = \langle Y^{real}, CX^{real} \rangle,
\]

\[
\langle (I + B_n^{real} C)X^{img}, CX^{img} \rangle + \langle B_n^{img} C X^{real}, CX^{img} \rangle = \langle Y^{img}, CX^{img} \rangle.
\]
On the Foldy-Lax approximation of the scattering by small bodies

By summing up (4.4) and (4.5) will give

\[ \langle X^{\text{real}}, CX^{\text{real}} \rangle + \langle B_n^{\text{real}} CX^{\text{real}}, CX^{\text{real}} \rangle + \langle X^{\text{img}}, CX^{\text{img}} \rangle + \langle B_n^{\text{real}} CX^{\text{img}}, CX^{\text{img}} \rangle = \langle Y^{\text{real}}, CX^{\text{real}} \rangle + \langle Y^{\text{img}}, CX^{\text{img}} \rangle. \] (4.6)

Indeed,

\[ \langle B_n^{\text{img}} CX^{\text{img}}, CX^{\text{real}} \rangle = \langle CX^{\text{img}}, B_n^{\text{img}} CX^{\text{real}} \rangle = \langle CX^{\text{img}}, B_n^{\text{img}} CX^{\text{real}} \rangle = \langle B_n^{\text{img}} CX^{\text{real}}, CX^{\text{img}} \rangle. \]

We can observe that, the right-hand side in (4.6) does not exceed

\[ \langle X^{\text{real}}, CX^{\text{real}} \rangle^{1/2} \langle Y^{\text{real}}, CY^{\text{real}} \rangle^{1/2} + \langle X^{\text{img}}, CX^{\text{img}} \rangle^{1/2} \langle Y^{\text{img}}, CY^{\text{img}} \rangle^{1/2} \leq 2 \langle X^{\text{real}}, CX^{\text{real}} \rangle^{1/2} \langle Y^{\text{real}}, CY^{\text{real}} \rangle^{1/2}. \] (4.7)

Here \( W_m^\text{real} + W_m^\text{img} \)^{1/2} = |W_m|, for \( W = X, Y \) and \( m = 1, \ldots, M \). Consider the second term in the left-hand side of (4.6). Using the mean value theorem for harmonic functions we deduce

\[ \langle B_n^{\text{real}} CX^{\text{real}}, CX^{\text{real}} \rangle = \sum_{1 \leq j, m \leq M, j \neq m} \Phi^{\text{real}}(z_m, z_j) \tilde{C}_j \tilde{C}_m X_j^{\text{real}} X_m^{\text{real}} \]

\[ \geq t \sum_{1 \leq j, m \leq M, j \neq m} \frac{\tilde{C}_j \tilde{C}_m X_j^{\text{real}} X_m^{\text{real}}}{|B(j)||B(m)|} \int_{B(j)} \int_{B(m)} \Phi_0(x, y) \, dx \, dy, \]

Similarly, if we consider the fourth term in the left-hand side of (4.6), we deduce

\[ \langle B_n^{\text{real}} CX^{\text{real}}, CX^{\text{real}} \rangle = \sum_{1 \leq j, m \leq M, j \neq m} \Phi^{\text{real}}(z_m, z_j) \tilde{C}_j \tilde{C}_m X_j^{\text{img}} X_m^{\text{img}} \]
Consider the first term in the right-hand side of (4.9), denote it by $D$ of all our small obstacles $\{z\}_{j=1}^{M}$ with the center at the origin.

Let $\Omega$ be a large ball with radius $R$. Also let $\Omega \subset \Omega$ be a ball with fixed radius $r \leq R$, which consists of all our small obstacles $D_m$ and also the balls $B^{(m)}$, for $m = 1, \ldots, M$.

Let $\mathcal{Y}^{\text{real}}(x)$ and $\mathcal{Y}^{\text{img}}(x)$ be piecewise constant functions defined on $\mathbb{R}^3$ as

$$
\mathcal{Y}^{\text{real(img)}}(x) = \begin{cases} 
X_m^{\text{real(img)}} C_m & \text{in } B^{(m)}, \quad m = 1, \ldots, M, \\
0 & \text{otherwise}.
\end{cases}
$$

Then

$$
(B_m^{\text{real}} X^{\text{real}}, C X^{\text{real}}) \geq \frac{36 t}{\pi^2 d^6} \left( \int_{\Omega} \int_{\Omega} \Phi_0(x, y) \mathcal{Y}^{\text{real}}(x) \mathcal{Y}^{\text{real}}(y) \, dx \, dy \right)
- \sum_{m=1}^{M} C_m^{2} X_m^{\text{real}} \int_{B^{(m)}} \int_{B^{(m)}} \Phi_0(x, y) \, dx \, dy
$$

$$
(B_m^{\text{real}} X^{\text{img}}, C X^{\text{img}}) \geq \frac{36 t}{\pi^2 d^6} \left( \int_{\Omega} \int_{\Omega} \Phi_0(x, y) \mathcal{Y}^{\text{img}}(x) \mathcal{Y}^{\text{img}}(y) \, dx \, dy \right)
- \sum_{m=1}^{M} C_m^{2} X_m^{\text{img}} \int_{B^{(m)}} \int_{B^{(m)}} \Phi_0(x, y) \, dx \, dy
$$

Applying the mean value theorem to the harmonic function $\frac{1}{4\pi|x-y|}$, as done in [16, p:109-110], we have the following estimate

$$
\int_{B^{(m)}} \int_{B^{(m)}} \Phi_0(x, y) \, dx \, dy \leq \frac{1}{4\pi} \int_{B_d} \int_{B_d} \frac{1}{|x-y|} \, dx \, dy \leq \frac{\pi d^5}{60}
$$

Consider the first term in the right-hand side of (4.9), denote it by $A^{\text{real}}_R$, then by Greens theorem

$$
A^{\text{real}}_R := \int_{\Omega} \int_{\Omega} \Phi_0(x, y) \mathcal{Y}^{\text{real}}(x) \mathcal{Y}^{\text{real}}(y) \, dx \, dy
$$

$$
= \int_{\Omega} \left| \nabla_x \int_{\Omega} \Phi_0(x, y) \mathcal{Y}^{\text{real}}(y) \, dy \right|^2 \, dx
- \int_{\Omega} \left( \frac{\partial}{\partial y_z} \int_{\Omega} \Phi_0(x, y) \mathcal{Y}^{\text{real}}(y) \, dy \right) \left( \int_{\Omega} \Phi_0(x, y) \mathcal{Y}^{\text{real}}(y) \, dy \right) \, dS_x.
$$

We have

$$
C^{\text{real}}_R = \int_{\Omega} \left( \int_{\Omega} \frac{\partial}{\partial y_z} \Phi_0(x, y) \mathcal{Y}^{\text{real}}(y) \, dy \right) \left( \int_{\Omega} \Phi_0(x, y) \mathcal{Y}^{\text{real}}(y) \, dy \right) \, dS_x
$$

$$
= \int_{\Omega} \left( \int_{\Omega} \frac{\partial}{\partial y_z} \Phi_0(x, y) \mathcal{Y}^{\text{real}}(y) \, dy \right) \left( \int_{\Omega} \Phi_0(x, y) \mathcal{Y}^{\text{real}}(y) \, dy \right) \, dS_x
$$

$$
= \int_{\Omega} \left( \int_{\Omega} \frac{1}{4\pi|x-y|^3} \mathcal{Y}^{\text{real}}(y) \, dy \right) \left( \int_{\Omega} \frac{1}{4\pi|x-y|^3} \mathcal{Y}^{\text{real}}(y) \, dy \right) \, dS_x.
$$
which gives the following estimate;

\[ |C_R^{real}| \leq \frac{1}{16\pi^2} \int_{\partial \Omega} \frac{1}{|R - r|^3} \left( \int_{\Omega_x} |T^{real}(y)| \, dy \right)^2 \, dS_x \]

\[ \leq \frac{1}{16\pi^2} \frac{(R - r)^3}{r^3} \int_{\partial \Omega} |\Omega_x| |T^{real}|^2 \, dS_x \]

\[ = \frac{12\pi(R - r)^3}{3} \sum_{m=1}^M \tan^2 C_m^2 |\Omega| \]

\[ = \frac{R^2r^3}{3(R - r)^3} \sum_{m=1}^M \tan^2 C_m^2. \]  

(4.14)

Substitution of (4.14) in (4.12) gives

\[ \int_{\Omega} \int_{\Omega} \Phi_0(x, y) T^{real}(x) T^{real}(y) \, dx \, dy \geq \int_{\Omega} \left| \nabla_x \int_{\Omega} \Phi_0(x, y) T^{real}(y) \, dy \right|^2 \, dx - \frac{R^2r^3}{3(R - r)^3} \sum_{m=1}^M \tan^2 C_m^2. \]

(4.15)

By considering the first term in the right-hand side of (4.10), and following the same procedure as mentioned in (4.12), (4.13) and (4.14), we obtain

\[ \int_{\Omega} \int_{\Omega} \Phi_0(x, y) Y^{img}(x) Y^{img}(y) \, dx \, dy \geq \int_{\Omega} \left| \nabla_x \int_{\Omega} \Phi_0(x, y) Y^{img}(y) \, dy \right|^2 \, dx - \frac{R^2r^3}{3(R - r)^3} \sum_{m=1}^M \tan^2 C_m^2. \]

(4.16)

Under our assumption \( t \geq 0, (4.9), (4.10), (4.11), (4.15) \) and (4.16) lead to

\[ \langle B_n^{real} C X^{real}, C X^{real} \rangle \geq \frac{36t}{\pi^2 \, 3^6} \left( \int_{\Omega} \left| \nabla_x \int_{\Omega} \Phi_0(x, y) T^{real}(y) \, dy \right|^2 \, dx - \frac{R^2r^3}{3(R - r)^3} \sum_{m=1}^M \tan^2 C_m^2 \right), \]

(4.17)

\[ \langle B_n^{real} C X^{img}, C X^{img} \rangle \geq \frac{36t}{\pi^2 \, 3^6} \left( \int_{\Omega} \left| \nabla_x \int_{\Omega} \Phi_0(x, y) Y^{img}(y) \, dy \right|^2 \, dx - \frac{R^2r^3}{3(R - r)^3} \sum_{m=1}^M \tan^2 C_m^2 \right). \]

(4.18)

Then (4.6), (4.17) and (4.18) imply

\[ 1 - \frac{36t}{\pi^2 \, 3^6} \left[ \frac{R^2r^3}{3(R - r)^3} + \frac{\pi \, 6^5}{60} \right] \sum_{m=1}^M |X_m|^2 C_m \]

\[ = \left( 1 - \frac{36t}{\pi^2 \, 3^6} \left[ \frac{R^2r^3}{3(R - r)^3} + \frac{\pi \, 6^5}{60} \right] \max_{m=1}^M C_m \right) \sum_{m=1}^M |X_m|^2 C_m \]

\[ \leq \left( \sum_{m=1}^M \tan^2 C_m^2 \right)^{1/2} \left( \sum_{m=1}^M Y_m^{\max} C_m^2 \right)^{1/2} \left( \sum_{m=1}^M Y_m^{imag} C_m \right)^{1/2} \left( \sum_{m=1}^M Y_m^{imag} C_m \right)^{1/2} \]

\[ \leq 2 \left( \sum_{m=1}^M |X_m|^2 C_m \right)^{1/2} \left( \sum_{m=1}^M |Y_m|^2 C_m \right)^{1/2}. \]

(4.19)

As we have \( R \) arbitrary, by tending \( R \) to \( \infty \), we can write (4.19) as

\[ \left( 1 - \frac{36t}{5\pi \, 3^6} \right) \sum_{m=1}^M |X_m|^2 C_m \leq 2 \left( \sum_{m=1}^M |X_m|^2 C_m \right)^{1/2} \left( \sum_{m=1}^M |Y_m|^2 C_m \right)^{1/2}. \]

(4.20)
which yields
\[
\sum_{m=1}^{M} |X_m|^2 \bar{C}_m \leq 4 \left(1 - \frac{3t}{5\pi d} \max_{m=1}^{M} \bar{C}_m \right)^{-2} \sum_{m=1}^{M} |Y_m|^2 \bar{C}_m.
\]
Thus, if \( \max_{1 \leq m \leq M} \bar{C}_m < \frac{5\pi}{3} d \) and \( t \geq 0 \), then the matrix \( B \) in algebraic system (2.141) is invertible and the estimate (4.20) and so (2.143) holds.

REFERENCES