Abstract. We are concerned with the linearized, isotropic and homogeneous elastic scattering problem by many small rigid obstacles, Lipschitz regular, shapes in 3D case. We give a sufficient condition on the number of the obstacles, their sizes and the minimum distance between them under which the Foldy-Lax approximation is valid.

Key words. Elastic wave scattering, Small-scatterers, Foldy-Lax approximation, Capacitance.

AMS subject classifications. 35J08, 35Q61, 45Q05

1. Introduction and statement of the results. Let $B_1, B_2, \ldots, B_M$ be $M$ open, bounded and simply connected sets in $\mathbb{R}^3$ with Lipschitz boundaries, containing the origin. We assume that their sizes and Lipschitz constants are uniformly bounded. We set $D_m := \epsilon B_m + z_m$ to be the small bodies characterized by the parameter $\epsilon > 0$ and the locations $z_m \in \mathbb{R}^3$, $m = 1, \ldots, M$.

Assume that the Lamé coefficients $\lambda$ and $\mu$ are constants satisfying $\mu > 0$ and $3\lambda + 2\mu > 0$. Let $U^i$ be a solution of the Navier equation $(\Delta^e + \omega^2)U^i = 0$ in $\mathbb{R}^3$, $\Delta^e := (\mu \Delta + (\lambda + \mu) \nabla \text{div})$. We denote by $U^s$ the acoustic field scattered by the $M$ small bodies $D_m \subset \mathbb{R}^3$ due to the incident field $U^i$. We restrict ourselves to the scattering by rigid bodies. Hence the total field $U^t := U^i + U^s$ satisfies the following exterior Dirichlet problem of the elastic waves

$$
(\Delta^e + \omega^2)U^t = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \left( \bigcup_{m=1}^{M} D_m \right),
$$

$$
U^t|_{\partial D_m} = 0, \quad 1 \leq m \leq M
$$

with the Kupradze radiation conditions (K.R.C)

$$
\lim_{|x| \to \infty} |x|^\frac{d-2}{d} \left( \frac{\partial U_p}{\partial |x|} - i \kappa_p U_p \right) = 0, \quad \text{and} \quad \lim_{|x| \to \infty} |x|^\frac{d-2}{d} \left( \frac{\partial U_s}{\partial |x|} - i \kappa_s U_s \right) = 0,
$$

where the two limits are uniform in all the directions $\hat{x} := \frac{x}{|x|} \in \mathbb{S}^{d-1}$. Also, we denote $U_p := - \kappa_p^2 \nabla \cdot U^s$ to be the longitudinal (or the pressure or P) part of the field $U$ and $U_s := - \kappa_s^2 \nabla \times (\nabla \times U^s)$ to be the transversal (or the shear or S) part of the field $U^s$ corresponding to the Helmholtz decomposition $U^s = U_p + U_s$. The constants $\kappa_p := \frac{\omega}{c_p}$ and $\kappa_s := \frac{\omega}{c_s}$ are known as the longitudinal and transversal wavenumbers, $c_p := \sqrt{\lambda + 2\mu}$ and $c_s := \sqrt{\mu}$ are the corresponding phase velocities, respectively and $\omega$ is the frequency.

The scattering problem (1.1-1.3) is well posed in the Hölder and Sobolev spaces, see [12, 13, 15, 16] for instance, and the scattered field $U^s$ has the following asymptotic expansion:

$$
U^s(x) := \frac{e^{i\kappa_p |x|}}{|x|} U^s_p(\hat{x}) + \frac{e^{i\kappa_s |x|}}{|x|} U^s_s(\hat{x}) + O\left( \frac{1}{|x|^2} \right), \quad |x| \to \infty
$$

---

1RICAM, Austrian Academy of Sciences, Altenbergerstrasse 69, A-4040, Linz, Austria. (Email:durga.challa@oeaw.ac.at) Supported by the Austrian Science Fund (FWF): P22341-N18.

1RICAM, Austrian Academy of Sciences, Altenbergerstrasse 69, A-4040, Linz, Austria. (Email:mourad.sini@oeaw.ac.at) Partially supported by the Austrian Science Fund (FWF): P22341-N18.

3Let us recall that the surface $\partial B_j$ is of Lipschitz class with the Lipschitz constants $r_j, L_j > 0$ if for any $P \in \partial B_j$, there exists a rigid transformation of coordinates under which we have $P = 0$ and $B_j \cap B_j^0(0) = \{ x \in B_j(0) : x_3 > \varphi_j(x_1, x_2) \}$, $B_j^0(0)$ being the ball of center 0 and radius $r_j$, where $\varphi_j$ is a Lipschitz continuous function on the disc of center 0 and radius $r_j$, i.e. $B_j^0(0)$, satisfying $\varphi_j(0) = 0$ and $||\varphi_j||_{C^1(B_j^0(0))} \leq L_j$. 

---
uniformly in all directions \( \hat{x} \in S^{d-1} \). The longitudinal part of the far-field, i.e. \( U_p^\infty(\hat{x}) \) is normal to \( S^2 \) while the transversal part \( U_s^\infty(\hat{x}) \) is tangential to \( S^2 \). As usual in scattering problems we use plane incident waves in this work. For the Lamé system, the full plane incident wave is of the form \( U^i(x, \theta) := \alpha \theta e^{i\kappa c_0 \theta \cdot x} + \beta \theta^t e^{i\kappa c_0 \theta^t \cdot x} \), where \( \theta^t \) is any direction in \( S^2 \) perpendicular to the incident direction \( \theta \) in \( S^2 \), \( \alpha, \beta \) are arbitrary constants. In particular, the pressure and shear incident waves are \( U^p(x, \theta) := \theta e^{i\kappa c_0 \theta \cdot x} \) and \( U^{s,}(x, \theta) := \theta^t e^{i\kappa c_0 \theta^t \cdot x} \), respectively. Pressure incident waves propagate in the direction of \( \theta \), whereas shear incident waves propagate in the direction of \( \theta^t \). The functions \( U_p^\infty(\hat{x}, \theta) := U_p^\infty(\hat{x}) \) and \( U_s^\infty(\hat{x}, \theta) := U_s^\infty(\hat{x}) \) for \((\hat{x}, \theta) \in S^2 \times S^2 \) are called the P part and the S part of the far-field pattern respectively.

**Definition 1.1.** We define

1. \( a \) as the maximum among the diameters, \( \text{diam} \), of the small bodies \( D_m \), i.e.
   \[
a := \max_{1 \leq m \leq M} \text{diam}(D_m) \quad (= \epsilon \max_{1 \leq m \leq M} \text{diam}(B_m), \quad (1.5)
   \]

2. \( d \) as the minimum distance between the small bodies \( \{D_1, D_2, \ldots, D_m\} \), i.e.
   \[
d := \min_{m \neq j} d_{mj},
   \]
   where \( d_{mj} := \text{dist}(D_m, D_j) \).

3. \( \omega_{\text{max}} \) as the upper bound of the used frequencies, i.e. \( \omega \in [0, \omega_{\text{max}}] \).

4. \( \Omega \) to be a bounded domain in \( \mathbb{R}^3 \) containing the small bodies \( D_m \), \( m = 1, \ldots, M \).

The main result of this paper is the following theorem.

**Theorem 1.2.** There exist two positive constants \( a_0 \) and \( c_0 \) depending only on the size of \( \Omega \), the Lipschitz character of \( B_m \), \( m = 1, \ldots, M \), \( d_{\text{max}} \) and \( \omega_{\text{max}} \) such that if

\[
a \leq a_0 \quad \text{and} \quad \sqrt{M - \frac{a}{d}} \leq c_0 \quad (1.6)
\]

then the P-part, \( U_p^\infty(\hat{x}, \theta) \), and the S-part, \( U_s^\infty(\hat{x}, \theta) \), of the far-field pattern have the following asymptotic expressions

\[
U_p^\infty(\hat{x}, \theta) = -\frac{1}{4\pi c_p}(I \otimes \hat{x}) \left[ \sum_{m=1}^M e^{-i\frac{\kappa}{c_p} \cdot \hat{x} \cdot z_m} Q_m + O \left( Ma^2 + M(M - 1) \frac{a^3}{d^2} + M(M - 1)^2 \frac{a^4}{d^3} \right) \right],
\]

(1.7)

\[
U_s^\infty(\hat{x}, \theta) = \frac{1}{4\pi c_s}(I - \hat{x} \otimes \hat{x}) \left[ \sum_{m=1}^M e^{-i\frac{\kappa}{c_s} \cdot \hat{x} \cdot z_m} Q_m + O \left( Ma^2 + M(M - 1) \frac{a^3}{d^2} + M(M - 1)^2 \frac{a^4}{d^3} \right) \right].
\]

(1.8)

uniformly in \( \hat{x} \) and \( \theta \) in \( S^2 \). The constant appearing in the estimate \( O(.) \) depends only on the size of \( \Omega \), the Lipschitz character of the obstacles, \( a_0, c_0 \) and \( \omega_{\text{max}} \). The vector coefficients \( Q_m \), \( m = 1, \ldots, M \), are the solutions of the following linear algebraic system

\[
C_m^{-1} Q_m = -U^i(z_m, \theta) - \sum_{j=1, j \neq m}^M \Gamma^\omega(z_m, z_j) Q_j,
\]

(1.9)

for \( m = 1, \ldots, M \), with \( \Gamma^\omega \) denoting the Kupradze matrix of the fundamental solution to the Navier equation with frequency \( \omega \), \( C_m := \int_{\partial D_m} \sigma_m(s) ds \) and \( \sigma_m \) is the solution matrix of the integral equation of the first kind

\[
\int_{\partial D_m} \Gamma^0(s_m, s) \sigma_m(s) ds = I, \quad s_m \in \partial D_m,
\]

(1.10)
with \( I \) the identity matrix of order 3. The algebraic system (1.9) is invertible under the condition:

\[
\frac{a}{d} \leq c_1 t^{-1}
\]

with

\[
t := \left[ \frac{1}{c_p^2} - 2\text{diam}(\Omega) \frac{\omega}{c_p^3} \left( \frac{1 - \frac{1}{2} \kappa_{\nu} \text{diam}(\Omega)}{1 - \frac{1}{2} \kappa_{\nu} \text{diam}(\Omega)} \right)^{N_1} + \frac{1}{2^{N_1-1}} \right] - \text{diam}(\Omega) \frac{\omega}{c_p^3} \left( \frac{1 - \frac{1}{2} \kappa_{\nu} \text{diam}(\Omega)}{1 - \frac{1}{2} \kappa_{\nu} \text{diam}(\Omega)} \right) + \frac{1}{2^{N_1-1}}
\]

which is assumed to be positive\(^2\) and \( N_1 := [2\text{diam}(\Omega) \max \{\kappa_{\nu}, \kappa_{\rho}\} c^2] \), where \([\cdot]\) denotes the integral part and \( \ln c = 1 \). The constant \( c_1 \) depends only on the Lipschitz character of the reference bodies \( B_m \), \( m = 1, \ldots, M \).

If the second condition of (1.6) is replaced by the stronger one:

\[
(M - 1) \frac{a}{d} \leq c_2,
\]

then (1.7-1.8) is reduced to:

\[
U_p^\infty (\hat{x}, \theta) = \frac{1}{4\pi c_p^2} (\hat{x} \otimes \hat{x}) \sum_{m=1}^{M} e^{-i \frac{\omega}{c_p} (\hat{x} \cdot \hat{z})} Q_m + O \left( M a^2 + M (M - 1) \frac{a^3}{d^2} \right),
\]

\[
U_s^\infty (\hat{x}, \theta) = \frac{1}{4\pi c_s^2} (I - \hat{x} \otimes \hat{x}) \sum_{m=1}^{M} e^{-i \frac{\omega}{c_s} (\hat{x} \cdot \hat{z})} Q_m + O \left( M a^2 + M (M - 1) \frac{a^3}{d^2} \right)
\]

and the algebraic system (1.9) is invertible.

The type of asymptotic expansions provided in Theorem 1.2 are useful for at least two reasons.

First, to estimate approximately the far-field, one needs only to compute the constant vectors \( Q_m \), which are solutions of a linear algebraic system, i.e. (1.9). This reduces considerably the computational effort comparing it to the methods based on integral equations, for instance, especially for a large number of obstacles. If the number of obstacles is actually very large then these asymptotics suggest the kind of effective medium that can produce the same far-fields and provides the error rate between the fields generated by the obstacles and those generated by the effective medium.

Second, using formulas of the type (1.7) and (1.8), one can solve the inverse problems which consists of localizing the centers, \( z_m \), of the obstacles from the far-field measurements using MUSIC type algorithm, for instance, and also estimating their sizes from the computed capacitances \( C_m \).

As a first reference on this topic, we mention the book by P. Martin [18] where the multiple scattering issue is well discussed and documented in its different scales. When the obstacles are distributed periodically in the whole domain, then homogenization techniques apply, see for instance [8,14,17]. As we see it in the previous theorem, we assume no periodicity. For such media, the type of result presented here are known, for the acoustic and electromagnetic models, in a series of works by A. Ramm, see [28–30] and the references therein for his recent related results. However, he used the (rough) condition \( \frac{a}{d} \ll 1 \) and no explicit mention has been made on the number of obstacles \( M \). Recently, in [11], the authors derived such approximation errors under a quite general condition on the denseness of the scatterers (i.e. involving \( M, a \) and \( d \)), i.e. of the form (1.6). The analysis is based on the use of integral equation methods and in particular the precise scaling of the surface layer potential operators. The goal of the present work is to extend those results to the Lamé system.

\(^2\)If, in particular, \( \text{diam}(\Omega) \max \{\kappa_{\nu}, \kappa_{\rho}\} c^2 < 1 \), then \( N_1 = 1 \) and hence \( t = \left[ \frac{1}{c_p^2} - 4\text{diam}(\Omega) \left( \frac{\omega}{c_p^3} \right)^2 \right] \). Assuming the Lamé coefficient \( \lambda \) to be positive, then \( c_p > c_s \). Hence, in this case, if \( \Omega \) is such that \( \text{diam}(\Omega) < \frac{c_s^3}{c_p^3} \) then \( t > 0 \).
The integral equation methods are widely used in such a context, see for instance the series of works by H. Ammari and H. Kang and their collaborators, as [3] and the references therein. The difference between their asymptotic expansion and the one described in the previous theorem is that their polarization tensors are build up from densities which are solutions of a system of integral equations while in the previous theorem the approximating terms are build up from the linear algebraic system (1.9). Due to motivations coming from inverse problems, apart from few works as [4], they consider well separated scatterers and hence their asymptotic expansions are given only in terms of the size of the scatterers. Regarding the Lamé system, we cite their works [2, 5–7] where, as we just mentioned, the asymptotics are given in terms of the size of the scatterers only. It is worth mentioning, however, that in these works, the authors considered transmission problems and showed that the corresponding moment tensors are in general anisotropic. If the inclusions are spherical, including the extreme cases of soft or rigid inclusions under certain conditions on the Lamé parameters, then these moment tensors are isotropic.

Let us mention the variational approach by V. Maz’ya, A. Movchan and M. Nieves [20, 23] where they study the Poisson problem and obtain estimates in forms similar to the previous theorem with weaker conditions on the Lamé parameters, then these moment tensors are isotropic.

Consider now the special case $d = a^t$, $M = a^{-s}$ with $t, s > 0$. Then the asymptotic expansions (1.15-1.16) can be rewritten as

$$U_p^\infty(\hat{x}, \theta) = \frac{1}{4\pi c_p} (\hat{x} \otimes \hat{x}) \left[ \sum_{m=1}^{M} e^{-\frac{\text{ir}^m}{c_p} \hat{x} \cdot \hat{z} m} Q_m \right]$$

$$+ O \left( M \left[ a^2 + a^3 \frac{d^3-3\alpha}{d^3-6\alpha} + a^4 \frac{d^3-6\alpha}{d^3-2\alpha} \right] + M(M-1) \left[ \frac{a^3}{d^2\alpha} + \frac{a^4}{d^3-\alpha} + \frac{a^4}{d^3-2\alpha} \right] + M(M-1)^2 \frac{a^4}{d^3\alpha} \right),$$

(1.15)

$$U_s^\infty(\hat{x}, \theta) = \frac{1}{4\pi c_s^2} (I - \hat{x} \otimes \hat{x}) \left[ \sum_{m=1}^{M} e^{-\frac{\text{ir}^m}{c_s} \hat{x} \cdot \hat{z} m} Q_m \right]$$

$$+ O \left( M \left[ a^2 + a^3 \frac{d^3-3\alpha}{d^3-6\alpha} + a^4 \frac{d^3-6\alpha}{d^3-2\alpha} \right] + M(M-1) \left[ \frac{a^3}{d^2\alpha} + \frac{a^4}{d^3-\alpha} + \frac{a^4}{d^3-2\alpha} \right] + M(M-1)^2 \frac{a^4}{d^3\alpha} \right).$$

(1.16)

where $0 < \alpha \leq 1$.

Consider now the special case $d = a^t$, $M = a^{-s}$ with $t, s > 0$. Then the asymptotic expansions (1.15-1.16) can be rewritten as

$$U_p^\infty(\hat{x}, \theta) = \frac{1}{4\pi c_p} (\hat{x} \otimes \hat{x}) \left[ \sum_{m=1}^{M} e^{-\frac{\text{ir}^m}{c_p} \hat{x} \cdot \hat{z} m} Q_m \right]$$

$$+ O \left( a^{2-s} + a^{3-s} \frac{d^3-5\alpha+3\alpha}{d^3-6\alpha} + a^{4-s} \frac{d^3-9\alpha+6\alpha}{d^3-2\alpha} \right. + a^{3-2s} \frac{d^3-3\alpha}{d^3-2\alpha} + a^{4-3s} \frac{d^3-3\alpha}{d^3-2\alpha} + a^{4-2s} \frac{d^3-5\alpha+2\alpha}{d^3-2\alpha} \right),$$

(1.17)
The Foldy-Lax approximation of the elastic scattering by many small bodies

\[ U_s^\infty(\hat{x}, \theta) = \frac{1}{4\pi c_s^2} (I - \hat{x} \otimes \hat{x}) \left[ \sum_{m=1}^M e^{-i\hat{x} \cdot \hat{z}_m} Q_m + O \left( a^2 + a^3 - a - a^4 + a^5 \right) \right] \]

As the diameter d tends to zero the error term tends to zero for \( t \) and \( s \) such that \( 0 < t < 1 \) and \( 0 < s < \min\{2(1-t), \frac{12-9t}{4}, \frac{20-15t}{12}, \frac{4}{3} - t \alpha \} \). In particular for \( t = \frac{1}{3}, s = 1 \), we have

\[ U_p^\infty(\hat{x}, \theta) = \frac{1}{4\pi c_p^2} (\hat{x} \otimes \hat{x}) \left[ \sum_{m=1}^M e^{-i\hat{x} \cdot \hat{z}_m} Q_m + O \left( a + a^2 + a^3 - a + a^4 \right) \right], \]

\[ U_s^\infty(\hat{x}, \theta) = \frac{1}{4\pi c_s^2} (I - \hat{x} \otimes \hat{x}) \left[ \sum_{m=1}^M e^{-i\hat{x} \cdot \hat{z}_m} Q_m + O \left( a^2 \right) \right], \quad \text{[obtained for } \alpha = \frac{1}{4} \text{]} \quad (1.17) \]

\[ U_s^\infty(\hat{x}, \theta) = \frac{1}{4\pi c_s^2} (I - \hat{x} \otimes \hat{x}) \left[ \sum_{m=1}^M e^{-i\hat{x} \cdot \hat{z}_m} Q_m + O \left( a^2 \right) \right], \quad \text{[obtained for } \alpha = \frac{1}{4} \text{]} \quad (1.18) \]

This particular case can be used to derive the effective medium by perforation using many small bodies, see [29, 30] for the acoustic models for instance. The result of Remark 1.3, in particular (1.17) and (1.18), ensures the rate of the error in deriving such an effective medium.

2. Proof of Theorem 1.2. We wish to warn the reader that in our analysis we use sometimes the parameter \( \epsilon \) and some other times the parameter \( a \) as they appear naturally in the estimates. But we bear in mind the relation (1.5) between \( \epsilon \) and \( a \).

2.1. The fundamental solution. The Kupradze matrix \( \Gamma^\omega = (\Gamma^\omega_{ij})_{i,j=1}^3 \) of the fundamental solution to the Navier equation is given by

\[ \Gamma^\omega(x, y) = \frac{1}{\mu} \Phi_{\kappa, \omega}(x, y) I + \frac{1}{\omega^2} \nabla_x \nabla_y^\top [\Phi_{\kappa, \omega}(x, y) - \Phi_{\kappa, \omega}(x, y)], \quad (2.1) \]

where \( \Phi_{\kappa}(x, y) = \frac{1}{4\pi} \exp(i\kappa|x-y|) \) denotes the free space fundamental solution of the Helmholtz equation \( \Delta + \kappa^2 u = 0 \) in \( \mathbb{R}^3 \). The asymptotic behavior of Kupradze tensor at infinity is given as follows

\[ \Gamma^\omega(x, y) = \frac{1}{4\pi c_p^2} (I - \hat{x} \otimes \hat{x}) e^{i\kappa \omega|x|} + \frac{1}{4\pi c_s^2} (I - \hat{x} \otimes \hat{x}) e^{i\kappa \omega|x|} + O(|x|^{-2}) \quad (2.2) \]
with \( \hat{x} = \frac{x}{|x|} \in S^2 \) and I being the identity matrix in \( \mathbb{R}^3 \), see [1] for instance. As mentioned in [6], (2.1) can also be represented as

\[
\Gamma^\omega(x, y) = \frac{1}{4\pi} \sum_{l=0}^{\infty} \frac{i^l}{l!(l+2)} \frac{1}{\omega^2} (l+1) \kappa_x^{l+2} + \kappa_y^{l+2}) |x-y|^{l-1} I
- \frac{1}{4\pi} \sum_{l=0}^{\infty} \frac{i^l}{l!(l+2)} \frac{(l-1)}{\omega^2} (\kappa_x^{l+2} - \kappa_y^{l+2}) |x-y|^{l-3} (x-y) \otimes (x-y),
\]

from which we can get the gradient

\[
\nabla_y \Gamma^\omega(x, y) = -\frac{1}{4\pi} \sum_{l=0}^{\infty} \frac{i^l}{l!(l+2)} \frac{(l-1)}{\omega^2} \left[ (l+1) \kappa_x^{l+2} + \kappa_y^{l+2} \right] |x-y|^{l-3} (x-y) \otimes I
- (\kappa_x^{l+2} - \kappa_y^{l+2}) |x-y|^{l-3} (l-3) |x-y|^{-2} \otimes^3 (x-y) + I \otimes (x-y) + (x-y) \otimes I \right].
\]

2.2. The representation via double layer potential. We start with the following proposition on the solution of the problem (1.1-1.3) via the method of integral equations.

**Proposition 2.1.** For \( m = 1, 2, \ldots, M \), there exists \( \sigma_m \in H^r(\partial D_m) \), \( r \in [0, 1] \) such that the solution of the problem (1.1-1.3) is of the form

\[
U^m(t) = U^m(x) + \sum_{m=1}^{M} \int_{\partial D_m} \frac{\partial \Gamma^\omega(x, s)}{\partial \nu_m(s)} \sigma_m(s) ds, x \in \mathbb{R}^3 \setminus \left( \bigcup_{m=1}^{M} \mathcal{D}_m \right),
\]

where \( \frac{\partial}{\partial \nu_m(\cdot)} \) denotes the conformal derivative on \( \partial D_m \) and is defined as

\[
\frac{\partial}{\partial \nu_m(\cdot)} := \lambda (\text{div} \cdot) N_m + \mu (\nabla \cdot + \nabla^\top) N_m \text{ on } \partial D_m,
\]

where \( N_m \) is the outward unit normal vector of \( \partial D_m \).

**Proof of Proposition 2.1.** We look for the solution of the problem (1.1-1.3) of the form (2.5), then from the Dirichlet boundary condition (1.2) and the jumps of the double layer potentials, we obtain

\[
\frac{\sigma_j(s_j)}{2} + \int_{\partial D_j} \frac{\partial \Gamma^\omega(s_j, s)}{\partial \nu_j(s)} \sigma_j(s) ds + \sum_{m=1}^{M} \int_{\partial D_m} \frac{\partial \Gamma^\omega(s_j, s)}{\partial \nu_m(s)} \sigma_m(s) ds = -U^m(s_j), \forall s_j \in \partial D_j, j = 1, \ldots, M.
\]

One can write the system (2.7) in a compact form as \((\frac{1}{2} I + DL + DK) \sigma = -U^m \text{ with } I := (I_{mj})_{m,j=1}^M,\)

\[
DL := (D_{mj})_{m,j=1}^M \text{ and } DK := (DK_{mj})_{m,j=1}^M,
\]

where

\[
I_{mj} = \begin{cases} I, & \text{Identity operator} \\ 0, & \text{zero operator} \end{cases}, \quad DL_{mj} = \begin{cases} D_{mj}, & m = j \\ 0, & \text{else} \end{cases}, \quad DK_{mj} = \begin{cases} D_{mj}, & m \neq j \\ 0, & \text{else} \end{cases}
\]

\[
U^m = (U^1, \ldots, U^M)^T \text{ and } \sigma = (\sigma_1, \ldots, \sigma_M)^T.
\]

Here, for the indices \( m \) and \( j \) fixed, \( D_{mj} \) is the integral operator

\[
D_{mj}(\sigma_j(t)) := \int_{\partial D_j} \frac{\partial \Gamma^\omega(t, s)}{\partial \nu_j(s)} \sigma_j(s) ds.
\]

The operator \( \frac{1}{2} I + DL_{mm} : H^r(\partial D_m) \rightarrow H^r(\partial D_m) \) is Fredholm with zero index and for \( m \neq j \), \( D_{mj} : H^r(\partial D_j) \rightarrow H^r(\partial D_m) \) is compact for \( 0 \leq r \leq 1 \), when \( \partial D_m \) has a Lipschitz regularity, see [19,25,26].

\[3\text{In [19,25,26], this property is proved for the case } \omega = 0. \text{ By a perturbation argument, we have the same results for every } \omega \text{ in } [0, \omega_{\text{max}}], \text{ assuming that } \omega_{\text{max}} \text{ is smaller than the first eigenvalue } w_{0,j} \text{ of the Dirichlet-Lamé operator in } D_{mj}. \text{ By a comparison theorem, see [21, (6.131) in Lemma 6.3.6] for instance, we know that } \mu w_L \leq w_{0,j} \text{ where } w_L \text{ is the first eigenvalue of the Dirichlet-Laplacian operator in } D_{mj}. \text{ Now, we know that } \left( \frac{1}{\nu} \sqrt{\frac{\omega_{\text{max}}}{\omega_{\text{max}}}} \right)^2 \leq w_L. \text{ Then, we need } \omega_{\text{max}} < \sqrt{2} \frac{\omega_{\text{max}}}{\sqrt{\frac{\omega_{\text{max}}}{\omega_{\text{max}}}}} \text{ which is satisfied if } a < \sqrt{\frac{\omega_{\text{max}}}{\omega_{\text{max}}}} \sqrt{\frac{\omega_{\text{max}}}{\omega_{\text{max}}}}. \text{ Here } j_{1/2}/j_{1/2} \text{ is the 1st positive zero of the Bessel function } J_{1/2}.\]
Similarly, we conclude then that we have the following representation of \( s \):

\[
U(x) = \sum_{m=1}^{M} \int_{\partial D_m} \frac{\partial \Gamma(s, s)}{\partial v_m(s)} \sigma_m(s) ds, \quad \text{in } \mathbb{R}^3 \setminus \left( \bigcup_{m=1}^{M} \partial D_m \right)
\]

and

\[
\tilde{U}(x) = \sum_{m=1}^{M} \int_{\partial D_m} \frac{\partial \Gamma(s, s)}{\partial v_m(s)} \sigma_m(s) ds, \quad \text{in } \bigcup_{m=1}^{M} D_m.
\]

Then \( \tilde{U} \) satisfies \( \Delta^2 \tilde{U} + \omega^2 \tilde{U} = 0 \) for \( x \in \mathbb{R}^3 \setminus \left( \bigcup_{m=1}^{M} \partial D_m \right) \), with K.R.C and \( \tilde{U}(x) = 0 \) on \( \bigcup_{m=1}^{M} \partial D_m \).

Similarly, \( \tilde{U} \) satisfies \( \Delta^2 \tilde{U} + \omega^2 \tilde{U} = 0 \) for \( x \in \bigcup_{m=1}^{M} D_m \) with \( \tilde{U}(x) = 0 \) on \( \bigcup_{m=1}^{M} \partial D_m \). Taking the trace on \( \partial D_m \), \( m = 1, \ldots, M \),

\[
\tilde{U}(s) = 0 \implies \mathcal{D}_{mm}(\sigma_m(s)) = 0 \quad \text{for } s \in \partial D_m \text{ and for } m = 1, \ldots, M.
\]

Difference between (2.10) and (2.11) implies that, \( \sigma_m = 0 \) for all \( m \).

We conclude then that \( \frac{1}{2} I + DL + DK \) is invertible since it is Fredholm of index zero and injective. This implies that \( \sigma = (\frac{1}{2} I + DL + DK)^{-1} U^{In} \).

\[
\sigma = (\frac{1}{2} I + DL + DK)^{-1} U^{In}
\]

The operator \( \frac{1}{2} I + DL \) is invertible since it is Fredholm of index zero and injective. This implies that

\[
||\sigma|| \leq \frac{||([\frac{1}{2} I + DL + DK]^{-1}|| U^{In} || ||DK||} {1 - ||([\frac{1}{2} I + DL]^{-1}|| ||DK||}.
\]

Here,

\[
||DK|| := ||DK||_{\mathcal{L} \left( \prod_{m=1}^{M} L^2(\partial D_m), \prod_{m=1}^{M} L^2(\partial D_m) \right)}
\]
have a single obstacle and then in the second step we deal with the multiple obstacle case.

\[ \Phi \]

We have the following lemma from \cite{11}.

\[ \| T \| \leq \frac{1}{2} \| I + DL \|^{-1} \]

\[ \| (I + DL)^{-1} \| \leq \frac{1}{2} \| I + DL \|^{-1} \]

\[ \| f \| \leq \| f \| \frac{1}{2} \| I + DL \|^{-1} \]

\[ \| \phi \| \leq \| \phi \| \frac{1}{2} \| I + DL \|^{-1} \]

In the following proposition, we provide conditions under which \( |L^{-1}| \| K | < 1 \) and then estimate \( \| \sigma \| \) via (2.13).

**Proposition 2.2.** There exists a constant \( c \) depending only on the size of \( \Omega \), the Lipschitz character of \( B_m, m = 1, \ldots, M, d_{\text{max}}, \) and \( \omega_{\text{max}} \) such that if

\[ \sqrt{M - 1} \epsilon < cd, \text{then} \| \sigma_m \|_{L^2(\partial D_m)} \leq c \epsilon \]

where \( c \) is a positive constant depending only on the Lipschitz character of \( B_m \).

**Proof of Proposition 2.2.**

For any functions \( f, g \) defined on \( \partial D_c \) and \( \partial B \) respectively, we define

\[ \hat{f}(\xi) := f(\epsilon\xi + z) \text{ and } \hat{g}(x) := g\left(\frac{x - z}{\epsilon}\right). \]

Let \( T_1 \) and \( T_2 \) be an orthonormal basis for the tangent plane to \( \partial D_c \) at \( x \) and let \( \partial/\partial T = \sum_{p=1}^{2} \partial/ \partial T_p \) \( T_p \), denote the tangential derivative on \( \partial D_c \). Then the space \( H^1(\partial D_c) \) is defined as

\[ H^1(\partial D_c) := \{ \phi \in L^2(\partial D_c); \partial \phi / \partial T \in L^2(\partial D_c) \}. \]

We have the following lemma from \cite{11}.

**Lemma 2.3.** Suppose \( 0 < \epsilon \leq 1 \) and \( D_c := \epsilon B + z \subset \mathbb{R}^n \). Then for every \( \psi \in L^2(\partial D_c) \) and \( \phi \in H^1(\partial D_c) \), we have

\[ \| \psi \|_{L^2(\partial D_c)} = \epsilon^{n-1} \| \hat{\psi} \|_{L^2(\partial B)} \]

and

\[ \epsilon^{n-1} \| \hat{\phi} \|_{H^1(\partial B)} \leq \| \phi \|_{H^1(\partial D_c)} \leq \epsilon^{n-1} \| \hat{\phi} \|_{H^1(\partial B)}. \]

We divide the rest of the proof of Proposition 2.2 into two steps. In the first step, we assume we have a single obstacle and then in the second step we deal with the multiple obstacle case.
2.3.1. The case of a single obstacle. Let us consider a single obstacle \( D_\epsilon := \epsilon B + z \). Then define the operator \( \mathcal{D}_{D_\epsilon} : L^2(\partial D_\epsilon) \to L^2(\partial D_\epsilon) \) by

\[
(\mathcal{D}_{D_\epsilon} \psi)(s) = \int_{\partial D_\epsilon} \frac{\partial \Gamma^\omega(s, t)}{\partial \nu(t)} \psi(t) dt.
\]

Following the arguments in the proof of Proposition 2.1, the integral operator \( \frac{1}{2} I + \mathcal{D}_{D_\epsilon} : L^2(\partial D_\epsilon) \to L^2(\partial D_\epsilon) \) is invertible. If we consider the problem (1.1-1.3) in \( \mathbb{R}^3 \setminus D_\epsilon \), we obtain

\[
\sigma = (\frac{1}{2} I + \mathcal{D}_{D_\epsilon})^{-1} U^\lambda, \quad \text{where } DL + DK =: \mathcal{D}_{D_\epsilon}
\]

and then

\[
||\sigma||_{L^2(\partial D_\epsilon)} \leq ||(\frac{1}{2} I + \mathcal{D}_{D_\epsilon})^{-1}||_{\mathcal{L}(L^2(\partial D_\epsilon), L^2(\partial D_\epsilon))} ||U^\lambda||_{L^2(\partial D_\epsilon)}.
\]

**Lemma 2.4.** Let \( \phi, \psi \in L^2(\partial D_\epsilon) \). Then,

\[
\mathcal{D}_{D_\epsilon} \phi = (\mathcal{D}_B^* \phi)^\vee,
\]

\[
\left(\frac{1}{2} I + \mathcal{D}_{D_\epsilon}\right) \psi = \left(\frac{1}{2} I + \mathcal{D}_B^* \right)^\vee \psi,
\]

\[
\left(\frac{1}{2} I + \mathcal{D}_{D_\epsilon}\right)^{-1} \phi = \left(\frac{1}{2} I + \mathcal{D}_B^* \right)^{-1} \phi
\]

\[
\left\|\left(\frac{1}{2} I + \mathcal{D}_{D_\epsilon}\right)^{-1}\right\|_{\mathcal{L}(L^2(\partial D_\epsilon), L^2(\partial D_\epsilon))} = \left\|\left(\frac{1}{2} I + \mathcal{D}_B^* \right)^{-1}\right\|_{\mathcal{L}(L^2(\partial B), L^2(\partial B))}
\]

and

\[
\left\|\left(\frac{1}{2} I + \mathcal{D}_{D_\epsilon}\right)^{-1}\right\|_{\mathcal{L}(H^1(\partial D_\epsilon), H^1(\partial D_\epsilon))} \leq \epsilon^{-1} \left\|\left(\frac{1}{2} I + \mathcal{D}_B^* \right)^{-1}\right\|_{\mathcal{L}(H^1(\partial B), H^1(\partial B))}
\]

with \( \mathcal{D}_B^* \psi(\xi) := \int_{\partial B} \frac{\partial \Gamma^\omega(\xi, \eta)}{\partial \nu(\eta)} \psi(\eta) d\eta \).

**Proof of Lemma 2.4.**

- We have,

\[
\mathcal{D}_{D_\epsilon} \psi(s) = \int_{\partial D_\epsilon} \frac{\partial \Gamma^\omega(s, t)}{\partial \nu(t)} \psi(t) dt
\]

\[
= \int_{\partial D_\epsilon} \left[ \lambda (\text{div}_t \Gamma^\omega(s, t)) N_t + \mu \left( \nabla_t \Gamma^\omega(s, t) + (\nabla_t \Gamma^\omega(s, t))^\top \right) N_t \right] \psi(t) dt
\]

\[
= \int_{\partial B} \epsilon^{-2} \left[ \lambda (\text{div}_\eta \Gamma^\omega(\xi, \eta)) N_\eta + \mu \left( \nabla_\eta \Gamma^\omega(\xi, \eta) + (\nabla_\eta \Gamma^\omega(\xi, \eta))^\top \right) N_\eta \right] \psi(\epsilon \eta + z) \epsilon^2 d\eta
\]

\[
= \int_{\partial B} \frac{\partial \Gamma^\omega(\xi, \eta)}{\partial \nu(\eta)} \psi(\epsilon \eta + z) d\eta
\]

\[
= \mathcal{D}_B^* \psi(\xi).
\]

The above gives us (2.22). From (2.22), we can obtain (2.23).
The following equalities
\[
\left( \frac{1}{2} I + \mathcal{D}_{D_r} \right) \left( \left( \frac{1}{2} I + \mathcal{D}'_{B_r} \right)^{-1} \phi \right) \overset{\text{(2.24)}}{=} \left( \frac{1}{2} I + \mathcal{D}'_{B} \right) \left( \frac{1}{2} I + \mathcal{D}'_{B} \right)^{-1} \phi = \phi^\nu = \phi
\]
provide us (2.24).

The following equalities
\[
\left\| \left( \frac{1}{2} I + \mathcal{D}_{D_r} \right)^{-1} \right\|_{L(L^2(\partial D_r), L^2(\partial D_r))} := \sup_{\phi(\neq 0) \in L^2(\partial D_r)} \frac{\left\| \left( \frac{1}{2} I + \mathcal{D}_{D_r} \right)^{-1} \phi \right\|_{L^2(\partial D_r)}}{\| \phi \|_{L^2(\partial D_r)}}
\]
\[
\overset{\text{(2.24)}}{=} \sup_{\phi(\neq 0) \in L^2(\partial D_r)} \epsilon \left\| \left( \frac{1}{2} I + \mathcal{D}_{D_r} \right)^{-1} \phi \right\|_{L^2(\partial B)} = \left\| \left( \frac{1}{2} I + \mathcal{D}'_{B} \right)^{-1} \right\|_{L(L^2(\partial B), L^2(\partial B))}
\]
provide us (2.25). By proceeding in the similar manner we can obtain (2.26) as mentioned below,
\[
\left\| \left( \frac{1}{2} I + \mathcal{D}_{D_r} \right)^{-1} \right\|_{L(H^1(\partial D_r), H^1(\partial D_r))} := \sup_{\phi(\neq 0) \in H^1(\partial D_r)} \frac{\left\| \left( \frac{1}{2} I + \mathcal{D}_{D_r} \right)^{-1} \phi \right\|_{H^1(\partial D_r)}}{\| \phi \|_{H^1(\partial D_r)}}
\]
\[
\overset{\text{(2.18),(2.19)}}{=} \sup_{\phi(\neq 0) \in H^1(\partial D_r)} \epsilon \left\| \left( \frac{1}{2} I + \mathcal{D}_{D_r} \right)^{-1} \phi \right\|_{H^1(\partial B)} = \epsilon^{-1} \left\| \left( \frac{1}{2} I + \mathcal{D}'_{B} \right)^{-1} \right\|_{L(H^1(\partial B), H^1(\partial B))}
\]

The next lemma provides us with an estimate of the left hand side of (2.25) by a constant \(C\) with a useful dependence of \(C\) in terms of \(B\) through its Lipschitz character and \(\omega\).

**Lemma 2.5.** The operator norm of \(\left( \frac{1}{2} I + \mathcal{D}_{D_r} \right)^{-1} : L^2(\partial D_r) \rightarrow L^2(\partial D_r)\) satisfies the estimate
\[
\left\| \left( \frac{1}{2} I + \mathcal{D}_{D_r} \right)^{-1} \right\|_{L(L^2(\partial D_r), L^2(\partial D_r))} \leq \mathcal{C}_0,
\]
with \(\mathcal{C}_0 := 4\pi \left\| \left( \frac{1}{2} I + \mathcal{D}'_{B} \right)^{-1} \right\|_{L(L^2(\partial B), L^2(\partial B))} \). Here, \(\mathcal{D}'_{B} : L^2(\partial B) \rightarrow L^2(\partial B)\) is the double layer potential with the wave number zero.

Here we should mention that if \(\epsilon^2 \leq \frac{4\pi}{\left( \frac{4\pi + 12\omega}{2\pi} + \frac{12\omega + 9\omega}{2\pi} \right)^2 \left( \frac{4\pi + 12\omega}{2\pi} \right)^2} \), then \(\mathcal{C}_0\) is bounded by \(\frac{4}{3} \left\| \left( \frac{1}{2} I + \mathcal{D}'_{B} \right)^{-1} \right\|_{L(L^2(\partial B), L^2(\partial B))} \), which is a universal constant depending only on \(\partial B\) through its Lipschitz character.
Proof of Lemma 2.5. To estimate the operator norm of \((\frac{1}{2}I + \mathcal{D}_{D_\omega})^{-1}\) we decompose \(\mathcal{D}_{D_\omega} = \mathcal{D}_{D_\omega}^L + \mathcal{D}_{D_\omega}^T\) into two parts \(\mathcal{D}_{D_\omega}^L\) (independent of \(\omega\)) and \(\mathcal{D}_{D_\omega}^T\) (dependent of \(\omega\)) given by

\[
\mathcal{D}_{D_\omega}^L \psi(s) := \int_{\partial D_\omega} \left( \frac{\partial}{\partial n(t)} \Gamma_0^0(s,t) \right) \psi(t) dt,
\]

\[
\mathcal{D}_{D_\omega}^T \psi(s) := \int_{\partial D_\omega} \left( \frac{\partial}{\partial n(t)} [\Gamma^\omega(s,t) - \Gamma_0^0(s,t)] \right) \psi(t) dt.
\]

With this definition, \(\frac{1}{2}I + \mathcal{D}_{D_\omega}^L : L^2(\partial D_\omega) \to L^2(\partial D_\omega)\) is invertible, see \([19, 25, 26]\). Hence, \(\frac{1}{2}I + \mathcal{D}_{D_\omega} = \left(\frac{1}{2}I + \mathcal{D}_{D_\omega}^L \right) \left( I + \left(\frac{1}{2}I + \mathcal{D}_{D_\omega}^L \right)^{-1} \mathcal{D}_{D_\omega}^T \right)\) and so

\[
\left\| \left( \frac{1}{2}I + \mathcal{D}_{D_\omega} \right)^{-1} \right\|_{\mathcal{L}(L^2(\partial D_\omega),L^2(\partial D_\omega))} = \left\| \left( I + \left(\frac{1}{2}I + \mathcal{D}_{D_\omega}^L \right)^{-1} \mathcal{D}_{D_\omega}^T \right)^{-1} \left(\frac{1}{2}I + \mathcal{D}_{D_\omega}^T \right) \right\|_{\mathcal{L}(L^2(\partial D_\omega),L^2(\partial D_\omega))} \leq \left\| \left( I + \left(\frac{1}{2}I + \mathcal{D}_{D_\omega}^L \right)^{-1} \mathcal{D}_{D_\omega}^T \right)^{-1} \right\|_{\mathcal{L}(L^2(\partial D_\omega),L^2(\partial D_\omega))} \times \left\| \left( \frac{1}{2}I + \mathcal{D}_{D_\omega}^T \right)^{-1} \right\|_{\mathcal{L}(L^2(\partial D_\omega),L^2(\partial D_\omega))}.
\]

So, to estimate the operator norm of \((\frac{1}{2}I + \mathcal{D}_{D_\omega})^{-1}\) one needs to estimate the operator norm of \(\left( I + \left(\frac{1}{2}I + \mathcal{D}_{D_\omega}^L \right)^{-1} \mathcal{D}_{D_\omega}^T \right)^{-1}\). In particular one needs to have the knowledge about the operator norms of \((\frac{1}{2}I + \mathcal{D}_{D_\omega}^L)^{-1}\) and \(\mathcal{D}_{D_\omega}^T\) to apply the Neumann series. For that purpose, we can estimate the operator norm of \((\frac{1}{2}I + \mathcal{D}_{D_\omega}^L)^{-1}\) from (2.25) by

\[
\left\| \left( \frac{1}{2}I + \mathcal{D}_{D_\omega}^L \right)^{-1} \right\|_{\mathcal{L}(L^2(\partial D_\omega),L^2(\partial D_\omega))} = \left\| \left( \frac{1}{2}I + \mathcal{D}_{B}^L \right)^{-1} \right\|_{\mathcal{L}(L^2(\partial B),L^2(\partial B))}.
\]

Here \(\mathcal{D}_{B}^L \hat{\psi}(\xi) := \int_{\partial B} \left( \frac{\partial}{\partial n(\eta)} \Gamma_0^0(\xi, \eta) \right) \hat{\psi}(\eta) d\eta\). From the definition of the operator \(\mathcal{D}_{D_\omega}^L\) in (2.29), for the kind of technique we used see (5.13), we deduce that

\[
\mathcal{D}_{D_\omega}^L \psi(s) = \int_{\partial B} \left( \frac{\partial}{\partial n(\eta)} [\Gamma^\omega(\xi, \eta) - \Gamma_0^0(\xi, \eta)] \right) \hat{\psi}(\eta) d\eta = \int_{\partial B} \left[ \lambda \left( \text{div}_\eta [\Gamma^\omega(\xi, \eta) - \Gamma_0^0(\xi, \eta)] \right) + \mu \left( \nabla_\eta [\Gamma^\omega(\xi, \eta) - \Gamma_0^0(\xi, \eta)] + (\nabla_\eta [\Gamma^\omega(\xi, \eta) - \Gamma_0^0(\xi, \eta)]^T) \right) \right] \hat{\psi}(\eta) d\eta
\]

where the vector \(I_1\) and the third order tensor \(I_2\) are estimated by using (2.3) and (2.4) as

\[
I_1 = \frac{-c^2}{4\pi} \sum_{l=2}^{\infty} \frac{c^{l-2}i^l}{l(l+2)} \frac{(l-1)}{\omega^2} \left[ -2\kappa_{\omega^2}^l + (l + 4)\kappa_{\mu^2}^l \right] |\xi - \eta|^{-3}(\xi - \eta),
\]

\[
I_2 = \frac{-c^2}{4\pi} \sum_{l=2}^{\infty} \frac{c^{l-2}i^l}{l(l+2)} \frac{(l-1)}{\omega^2} \left[ \left( (l+1)\kappa_{\omega^2}^l + \kappa_{\mu^2}^l \right) |\xi - \eta|^{-3}(\xi - \eta) \right. \left. \otimes I \right]
\]

\[
- \left( \kappa_{\omega^2}^l - \kappa_{\mu^2}^l \right) |\xi - \eta|^{-3} ((l-3)|\xi - \eta|^{-2} \otimes^3 (\xi - \eta) + I \otimes (\xi - \eta) + (\xi - \eta) \otimes I ) \right].
\]
Using the observation \( \| |x|^p \|_{L^2(D)} \leq \| |x|^p |D| \|_{L^2(D)} \) we obtain

\[
\left| \mathcal{D}^L_{\partial D_s} \psi(s) \right| \leq \frac{\lambda^2}{4\pi} \sum_{i=2}^{\infty} \frac{e^{-l^2/2}}{l(1 + 2)} \frac{(l - 1)}{\omega^2} \int_{\partial B} |\xi - \eta|^{l-2} |\psi(\eta)| d\eta + \\
2\mu \frac{e^{-l^2/2}}{4\pi} \sum_{i=2}^{\infty} \frac{e^{-l^2/2}}{l(1 + 2)} \frac{(l - 1)}{\omega^2} \int_{\partial B} |\xi - \eta|^{l-2} |\psi(\eta)| d\eta + \frac{(6\kappa_\omega + 4\kappa_\rho)}{8\omega^2} \int_{\partial B} |\psi(\eta)| d\eta
\]

\[
\leq \frac{\lambda^2}{4\pi} \left| \frac{\omega}{|2\partial B|} \right| \frac{1}{\omega^2} \left[ \sum_{i=2}^{\infty} \frac{e^{-l^2/2}}{l(1 + 2)} \frac{(l - 1)}{\omega^2} \left( 2\kappa_{\omega} + (l + 4)\kappa_{\rho} \right) |\xi - \cdot| \left[ L^2(\partial B) \right] \left| \partial B \right|^{1/2} \right] + \\
2\mu \frac{e^{-l^2/2}}{4\pi} \left| \frac{\omega}{|2\partial B|} \right| \frac{1}{\omega^2} \left[ \sum_{i=2}^{\infty} \frac{e^{-l^2/2}}{l(1 + 2)} \frac{(l - 1)}{\omega^2} \left( 2\kappa_{\omega} + (l + 4)\kappa_{\rho} \right) |\xi - \cdot| \left[ L^2(\partial B) \right] \left| \partial B \right|^{1/2} \right] + \frac{(\kappa_\omega + \kappa_\rho)}{4\omega^2}
\]

\[
\leq \omega^2 \frac{e^{-l^2/2}}{4\pi} \left| \frac{\omega}{|2\partial B|} \right| \frac{1}{\omega^2} \left[ \left( \frac{\mu}{\omega} + (\lambda + 4\mu) \sum_{i=0}^{\infty} \frac{1}{2} \left( \kappa_{\omega} \right) \left| \xi - \cdot \right| \left[ L^2(\partial B) \right] \left| \partial B \right|^{1/2} \right)^2 \right] + \\
\left( \frac{\mu}{\omega} + (3\lambda + 2\mu) \sum_{i=0}^{\infty} \frac{1}{2} \left( \kappa_{\rho} \right) \left| \xi - \cdot \right| \left[ L^2(\partial B) \right] \left| \partial B \right|^{1/2} \right)^2 \right) \right) + \frac{2 \min \{cs, cp\}}{\omega_{max} \max_m diam(B_m)}
\]

with \( \mathcal{C}_1 := \frac{|\partial B|^{1/2}}{4\pi} \left[ \frac{4\lambda + 17\mu}{2\omega^2} \right] + \frac{12\lambda + 9\mu}{2\omega^2} \]. From this we obtain,

\[
\left| \mathcal{D}^L_{\partial D_s} \psi(s) \right|_{L^2(\partial D_s)} = \int_{\partial D_s} \left| \mathcal{D}^L_{\partial D_s} \psi(s) \right| d\sigma \\
\leq \frac{2 \min \{cs, cp\}}{\omega_{max} \max_m diam(B_m)}
\]

\[
\left| \mathcal{D}^L_{\partial D_s} \psi \right|_{L^2(\partial D_s)} \leq \mathcal{C}_1 \omega^2 e^3 |\partial B|^{1/2} \left| \frac{\omega}{|2\partial B|} \right| \left| \psi \right|_{L^2(\partial B)}
\]

We estimate the norm of the operator \( \mathcal{D}^L_{\partial D_s} \) as

\[
\left| \mathcal{D}^L_{\partial D_s} \right|_{L^2(\partial D_s)} = \sup_{\psi(\emptyset) \in L^2(\partial D_s)} \left| \mathcal{D}^L_{\partial D_s} \psi \right|_{L^2(\partial D_s)} \left| \psi \right|_{L^2(\partial D_s)} \leq \mathcal{C}_1 \omega^2 e^3 |\partial B|^{1/2} \left| \frac{\omega}{|2\partial B|} \right| \left| \psi \right|_{L^2(\partial B)}
\]

\[
\left| \left( \frac{1}{2} I + \mathcal{D}^L_{\partial D_s} \right)^{-1} \right|_{L^2(\partial D_s)} \left| \mathcal{D}^L_{\partial D_s} \right|_{L^2(\partial D_s)} \left| \psi \right|_{L^2(\partial D_s)} \leq \left| \left( \frac{1}{2} I + \mathcal{D}^L_{\partial D_s} \right)^{-1} \right|_{L^2(\partial D_s)} \left| \mathcal{D}^L_{\partial D_s} \right|_{L^2(\partial D_s)} \left| \psi \right|_{L^2(\partial D_s)}
\]

(2.35)
By substituting the above and (2.31) in (2.30), we obtain the required result (2.27).

Assuming \( \epsilon \) to satisfy the condition \( \epsilon < \frac{1}{\sqrt{C_2\omega^2}} \), then \( \left\| \left( \frac{1}{2} I + D_{D_i}^{m_i} \right)^{-1} D_{D_i}^{m_i} \right\|_{L(L^2(\partial D_i), L^2(\partial D_i))} < 1 \) and hence by using the Neumann series we obtain the following bound

\[
\left\| \left( I + \frac{1}{2} I + D_{D_i}^{m_i} \right)^{-1} D_{D_i}^{m_i} \right\|_{L(L^2(\partial D_i), L^2(\partial D_i))} \leq \frac{1}{1 - \left\| \left( \frac{1}{2} I + D_{D_i}^{m_i} \right)^{-1} D_{D_i}^{m_i} \right\|_{L(L^2(\partial D_i), L^2(\partial D_i))}} \leq \frac{1}{1 - \hat{C}_3} := \frac{1}{1 - \hat{C}_2 \omega^2 \epsilon^2}.
\]

By substituting the above and (2.31) in (2.30), we obtain the required result (2.27).

**2.3.2. The multiple obstacle case.**

**Lemma 2.6.** For each \( k > 0 \) and for every \( n \in \mathbb{Z}^+ \) with \( n \geq k \varepsilon^2 \), \( n \geq k \varepsilon^2 \) [\( =: N(k) \)] we have \( n! \geq k^{n-1} \).

**Proof of Lemma 2.6.** The result is true for \( n = 1 \). The proof goes as follows for \( n > 1 \):

\[
n \geq k \varepsilon^2 \implies \ln k \leq \ln n - 2 \implies 4 \implies \ln k \leq \frac{\ln \sqrt{2\pi} - n}{n - 1} + \left( n + \frac{1}{2} \right) \ln n \implies (n - 1) \ln k \leq \ln \sqrt{2\pi} + \left( n + \frac{1}{2} \right) \ln n - n \implies k^{n-1} \leq \sqrt{2\pi} \frac{n}{e} \]

Now, we obtain the result using Stirling’s approximation \( n! \sim \sqrt{2\pi} \frac{n}{e} \), precisely \( \sqrt{2\pi} \frac{n}{e} \left( \frac{n}{e} \right)^n \leq n! \), see [31] for instance.

**Proposition 2.7.** For \( m, j = 1, 2, \ldots, M \), the operator \( D_{m_j} : L^2(\partial D_j) \to L^2(\partial D_m) \) defined in Proposition 2.1, see (2.9), enjoys the following estimates,

- For \( j = m \),

\[
\left\| \left( \frac{1}{2} I + \mathcal{D}_{m,m} \right)^{-1} \right\|_{L(L^2(\partial D_m), L^2(\partial D_m))} \leq \hat{C}_{6m},
\]

\[
\hat{C}_{6m} := \frac{4\pi \left\| \left( \frac{1}{2} I + \mathcal{D}_{m,m} \right)^{-1} \right\|_{L(L^2(\partial D_m), L^2(\partial D_m))}}{4\pi - \frac{2k+1}{2\varepsilon} + \frac{2k+3}{2\varepsilon} + \omega^2 \varepsilon^2 \left\| \mathcal{D}_{m,m} \right\|_{L(L^2(\partial D_m), L^2(\partial D_m))}}.
\]

\( ^4 \text{Since, } \frac{(n + \frac{1}{2})}{n - 1} > 1, \frac{\ln \sqrt{2\pi}}{n - 1} > 0 \) and \( 0 < \frac{n}{n - 1} < 2 \) for \( n > 1 \).
• For $j \neq m$,
\[
|\mathcal{D}_{mj}|_{L^2(\partial D_j), L^2(\partial D_m))} \leq \left[ \frac{\tilde{C}_7}{d^2} + \tilde{C}_8 \right] \frac{1}{4\pi} |\partial B| \epsilon^2,
\]
where
\[
|\partial B| := \max_m \partial B_m, \quad \tilde{C}_7 := \left( \frac{\lambda + 6\mu}{c^2_p} + \frac{2\lambda + 6\mu}{c^2_p} \right)
\]
and
\[
\tilde{C}_8 := \frac{\omega^2}{c^2_p} \left( \frac{\mu}{2} + (\lambda + 4\mu) \frac{1 - (\frac{1}{2}\kappa_{\omega'}\text{diam}(\Omega))}{1 - \frac{1}{2}\kappa_{\omega'}\text{diam}(\Omega)} \right) + \frac{\omega^2}{c^2_p} \left( \frac{\mu}{2} + (\lambda + 4\mu) \frac{1 - (\frac{1}{2}\kappa_{\rho'}\text{diam}(\Omega))}{1 - \frac{1}{2}\kappa_{\rho'}\text{diam}(\Omega)} \right)
\]
with $N_{\Omega} = [2\text{diam}(\Omega) \max\{\kappa_{\omega'}, \kappa_{\rho'}\}]^c$, where $[\cdot]$ denotes the integral part.

Proof of Proposition 2.7. The estimate (2.39) is nothing but (2.27) of Lemma 2.5, replacing $B$ by $B_m$, $z$ by $z_m$ and $D$ by $D_m$ respectively. It remains to prove the estimate (2.40). We have
\[
|\mathcal{D}_{mj}|_{L^2(\partial D_j), L^2(\partial D_m))} = \sup_{\psi(\neq 0) \in L^2(\partial D_j)} \frac{|\mathcal{D}_{mj}\psi|_{L^2(\partial D_j)}}{|\psi|_{L^2(\partial D_j)}}.
\]
Let $\psi \in L^2(\partial D_j)$ then for $s \in \partial D_m$, we have
\[
\mathcal{D}_{mj}\psi(s) = \int_{\partial D_j} \frac{\partial\Gamma^\omega(s,t)}{\partial n_j(t)} \psi(t)dt
\]
\[
= \int_{\partial D_j} [\lambda (\text{div} [\Gamma^\omega(s,t)]) N_t + \mu (\nabla_t [\Gamma^\omega(s,t)] + (\nabla_t [\Gamma^\omega(s,t)])^\top) N_t] \psi(t)dt
\]
\[
= \int_{\partial D_j} [\lambda I_{1'} \otimes N_t + \mu (I_{2'} + I_{2''}) N_t] \psi(t)dt,
\]
where the vector $I_{1'}$ and the third order tensor $I_{2'}$ are given by
\[
I_{1'} = -4\pi \sum_{l=0}^{\infty} \frac{l}{l(l+2)} \omega^2 \left[ 2\kappa_{\rho'}^{l+2} \right] |s-t|^{l-3} (s-t),
\]
\[
I_{2'} = -4\pi \sum_{l=2}^{\infty} \frac{l}{l(l+2)} \omega^2 \left[ ((l+1)\kappa_{\rho'}^{l+2} + \kappa_{\omega'}^{l+2}) |s-t|^{l-3} (s-t) \otimes I
\]
\[
- (\kappa_{\rho'}^{l+2} - \kappa_{\omega'}^{l+2}) |s-t|^{l-3} (l-3)(s-t)\otimes 3(s-t) + I \otimes (s-t) \otimes (s-t) \otimes I \right].
\]

Then, by using Lemma 2.6, we estimate
\[
|\mathcal{D}_{mj}\psi(s)| \leq \frac{\lambda}{4\pi} \left[ \frac{\kappa_{\rho'}^{2} + 2\kappa_{\omega'}^{2}}{\omega^2} \int_{\partial D_j} |s-t|^{-2} |\psi(t)|dt
\]
\[
+ \frac{1}{4\pi} \sum_{l=2}^{\infty} \frac{l-1}{l(l+2)} \omega^2 \left[ 2\kappa_{\rho'}^{l+2} + (l+4)\kappa_{\omega'}^{l+2} \right] \int_{\partial D_j} |s-t|^{l-2} |\psi(t)|dt
\]
\[
+ \frac{2\mu}{4\pi} \left[ \frac{3 \left( \kappa_{\rho'}^{2} + \kappa_{\omega'}^{2} \right)}{\omega^2} \int_{\partial D_j} |s-t|^{-2} |\psi(t)|dt + \frac{(6\kappa_{\rho'}^{2} + 4\kappa_{\omega'}^{2})}{8\omega^2} \int_{\partial D_j} |\psi(t)|dt
\]
\[
\frac{1}{\omega^2} + \sum_{l=2}^{\infty} \frac{1}{l!(l+2)} \frac{(l-1)}{\omega^2} (2\kappa_{s}^{l+2(2.45)\frac{\kappa_{p}^{l+2}}{\omega^2}
+ 4\omega^2 l_{m,j}\left(1 - \frac{1}{\omega^2} \right) (2\kappa_{s}^{l+2} + (l+4)\frac{\kappa_{p}^{l+2}}{\omega^2}) \text{diam}(\Omega)^{l-2}}
\right]
\]
\[
\leq \frac{\lambda}{4\pi} \frac{\|\psi\|_{L^2(\Omega)}}{\|\partial D_j\|} \frac{|\partial D_j|^2}{\omega^2} \left[ \frac{1}{d_{m,j}^2} \left( \frac{\lambda + 6\mu}{\omega s^2} + \frac{2\lambda + 6\mu}{\omega c_p^2} \right) + \frac{\mu}{2} \left( \frac{3\lambda + 4\mu}{2} \right) \sum_{l=0}^{N_{0} - 1} \left( \frac{1}{2} \kappa_{p} \text{diam}(\Omega) \right)^{l} + \frac{\mu}{2} \left( \frac{3\lambda + 4\mu}{2} \right) \sum_{l=0}^{N_{0} - 1} \left( \frac{1}{2} \kappa_{p} \text{diam}(\Omega) \right)^{l} \right]
\]
\[
\leq \frac{\epsilon \|\psi\|_{L^2(\Omega)}}{4\pi} \left[ \frac{1}{d_{m,j}^2} \left( \frac{\lambda + 6\mu}{\omega s^2} + \frac{2\lambda + 6\mu}{\omega c_p^2} \right) + \frac{\mu}{2} \left( \frac{3\lambda + 4\mu}{2} \right) \sum_{l=0}^{N_{0} - 1} \left( \frac{1}{2} \kappa_{p} \text{diam}(\Omega) \right)^{l} + \frac{\mu}{2} \left( \frac{3\lambda + 4\mu}{2} \right) \sum_{l=0}^{N_{0} - 1} \left( \frac{1}{2} \kappa_{p} \text{diam}(\Omega) \right)^{l} \right]
\]

form which, we get

\[
\|D_{m,j}\psi\|_{L^2(\partial D_m)} = \left( \int_{\partial D_m} |D_{m,j}\psi(s)|^2 \, ds \right)^{1/2} \leq \left( \frac{\epsilon}{4\pi} \right) \left[ \frac{\tilde{C}_7}{d_{m,j}} + \tilde{C}_8 \right] \frac{1}{4\pi} \frac{|\partial B_j|^2}{\|\psi\|_{L^2(\partial D_j)}} \left( \int_{\partial D_m} ds \right)^{1/2} \tag{2.45}
\]
\[= \left[ \frac{\tilde{C}_7}{d_{m}^2} + \tilde{C}_8 \right] \frac{1}{4\pi} \epsilon^2 |\partial B_j|^{\frac{1}{2}} |\partial B_m|^{\frac{1}{2}} \|\psi\|_{L^2(\partial D_j)}. \quad (2.46)\]

Substitution of (2.46) in (2.41) gives us
\[
||D_{m_j}||_{L^2(\partial D_j),L^2(\partial D_m)} \leq \left[ \frac{\tilde{C}_7}{d_{m}^2} + \tilde{C}_8 \right] \frac{1}{4\pi} \epsilon^2 |\partial B_j|^{\frac{1}{2}} |\partial B_m|^{\frac{1}{2}}
\leq \left[ \frac{\tilde{C}_7}{d^2} + \tilde{C}_8 \right] \frac{1}{4\pi} |\partial B| \epsilon^2.
\]

\[\square\]

End of the proof of Proposition 2.2. By substituting (2.39) in (2.15) and (2.40) in (2.14), we obtain
\[
\left\| \left( \frac{1}{2} I + DL \right)^{-1} \right\| \leq \frac{M}{\max_{m=1} \tilde{C}_m}
\quad (2.47)
\]
and
\[
||DK|| \leq \frac{M - 1}{4\pi} \left[ \frac{\tilde{C}_7}{d^2} + \tilde{C}_8 \right] |\partial B| \epsilon^2.
\quad (2.48)
\]

Hence, (2.48) and (2.47) jointly provide
\[
\left\| \left( \frac{1}{2} I + DL \right)^{-1} \right\| ||DK|| \leq \frac{M - 1}{4\pi} \left[ \frac{\tilde{C}_7}{d^2} + \tilde{C}_8 \right] \left( \max_{m=1} \tilde{C}_m \right) |\partial B| \epsilon^2,
\quad (2.49)
\]
By imposing the condition \( \left\| \left( \frac{1}{2} I + DL \right)^{-1} \right\| ||DK|| < 1 \), we get the following from (2.13) and (2.16)
\[
||\sigma_m||_{L^2(\partial D_m)} \leq ||\sigma|| \leq \frac{\left\| \left( \frac{1}{2} I + DL \right)^{-1} \right\| ||U||_{L^2(\partial D_m)}}{1 - \left\| \left( \frac{1}{2} I + DL \right)^{-1} \right\| ||DK||}
\leq \tilde{C}_p \left\| \left( \frac{1}{2} I + DL \right)^{-1} \right\| \left( \frac{M}{\max_{m=1}} ||U||_{L^2(\partial D_m)} \right) \left( \tilde{C}_p \geq \frac{1}{1 - \tilde{C}_a} \right)
\leq \frac{M}{\max_{m=1}} ||U||_{L^2(\partial D_m)} \left( \tilde{C} := \tilde{C}_p \max_{m=1} \tilde{C}_m \right)
\quad (2.50)
\]
for all \( m \in \{1, 2, \ldots, M\} \). But, for the plane incident wave of the Lamé system, \( U^i(x, \theta) := \alpha \theta e^{i\omega x / \epsilon} + \beta \theta e^{i\omega \sin \theta / \epsilon} \), we have
\[
||U||_{L^2(\partial D_m)} \leq (|\alpha| + |\beta|) \epsilon |\partial B_m|^{\frac{1}{2}} \leq (|\alpha| + |\beta|) \epsilon |\partial B|^{\frac{1}{2}}, \quad \forall m = 1, 2, \ldots, M.
\quad (2.51)
\]
Now by substituting (2.51) in (2.50), for each \( m = 1, \ldots, M \), we obtain
\[
||\sigma_m||_{L^2(\partial D_m)} \leq \tilde{C}(\omega) \epsilon,
\quad (2.52)
\]
where \( \tilde{C}(\omega) := \tilde{C} |\partial B|^{\frac{1}{2}} (|\alpha| + |\beta|) \).
The Foldy-Lax approximation of the elastic scattering by many small bodies

The condition \( \left\| \left( \frac{1}{2} I + DL \right)^{-1} \right\| \| DK \| < 1 \) is satisfied if \( \hat{C}_s < 1 \), i.e.

\[
\frac{M - 1}{4\pi} \left[ \hat{C}_7 \right. \left. \frac{d}{ds} + \hat{C}_8 \right] |\partial B| \left( \frac{\max_{m=1} \hat{C}_{bm}}{M} \right) \epsilon^2 < 1. \tag{2.53}
\]

The condition (2.53) reads as \( \sqrt{M - 1} \epsilon < \hat{c} \delta \) where we set

\[
\hat{c} := \frac{1}{4\pi} \left[ \hat{C}_7 + \hat{C} \delta d_{\text{max}}^2 \right] |\partial B| \left( \frac{\max_{m=1} \hat{C}_{bm}}{M} \right)^{-\frac{1}{2}}
\]

and it serves our purpose in Proposition 2.2 and hence in Theorem 1.2.

\[
\square
\]

2.4. The single layer potential representation and the total charge.

2.4.1. The single layer potential representation. For \( m = 1, 2, \ldots, M \), let \( U^{\sigma_m} \) be the solution of the problem

\[
\begin{align*}
\left\{ \begin{array}{ll}
(\Delta^e + \omega^2) U^{\sigma_m} = 0 & \text{in } D_m, \\
U^{\sigma_m} = \sigma_m & \text{on } \partial D_m.
\end{array} \right.
\end{align*} \tag{2.54}
\]

The function \( \sigma_m \) is in \( H^1(\partial D_m) \), see Proposition 2.1. Hence \( U^{\sigma_m} \in H^\frac{3}{2}(D_m) \) and then \( \frac{\partial U^{\sigma_m}}{\partial v_m} \bigg|_{\partial D_m} \in L^2(\partial D_m) \). From Proposition 2.1, the solution of the problem (1.1-1.3) has the form

\[
U^1(x) = U^1(x) + \sum_{m=1}^M \int_{\partial D_m} \Gamma^e(x, s) \frac{\partial U^{\sigma_m}(s)}{\partial v_m} ds, \quad x \in \mathbb{R}^3 \setminus \bigcup_{m=1}^M D_m. \tag{2.55}
\]

It can be written in terms of single layer potential using Gauss’s theorem as

\[
U^1(x) = U^1(x) + \sum_{m=1}^M \int_{\partial D_m} \Gamma^e(x, s) \frac{\partial U^{\sigma_m}(s)}{\partial v_m} ds, \quad x \in \mathbb{R}^3 \setminus \bigcup_{m=1}^M D_m. \tag{2.56}
\]

Indeed, by Betti’s third identity

\[
\int_{\partial D_m} \frac{\partial \Gamma^e(x, s)}{\partial v_m} \sigma_m(s) ds = \int_{\partial D_m} \Gamma^e(x, s) \frac{\partial U^{\sigma_m}(s)}{\partial v_m} ds + \int_{D_m} U^{\sigma_m}(s) \Delta^e \Gamma^e(x, s) ds - \int_{D_m} \Gamma^e(x, s) \Delta^e U^{\sigma_m}(s) ds.
\]

**Lemma 2.8.** For \( m = 1, 2, \ldots, M \), \( U^{\sigma_m} \), the solutions of the problem (2.54), satisfies the estimate

\[
\left\| \frac{\partial U^{\sigma_m}(s)}{\partial v_m} \right\|_{H^{-1}(\partial D_m)} \leq C_7, \tag{2.57}
\]

for some constant \( C_7 \) depending on \( B_m \), through its Lipschitz character but it is independent of \( \epsilon \).

**Proof of Lemma 2.8.** For \( m = 1, 2, \ldots, M \), we write

\[
U^m(x) := U^{\sigma_m}(\epsilon x + z_m), \forall x \in B_m.
\]

Then we obtain

\[
\begin{align*}
\left\{ \begin{array}{ll}
(\Delta^e + \epsilon^2 \omega^2) U^m(x) = \epsilon^2 (\Delta^e + \omega^2) U^{\sigma_m}(\epsilon x + z_m) = 0, & \text{for } x \in B_m, \\
U^m(\xi) = U^{\sigma_m}(\epsilon \xi + z_m) = \sigma(\epsilon \xi + z_m), & \text{for } \xi \in \partial B_m.
\end{array} \right.
\end{align*} \tag{2.58}
\]
and also \( \frac{\partial U^m}{\partial \nu_m}(\xi) = \nabla U^m(\xi) \cdot \nu_m(\xi) = \epsilon \nabla U^m(\epsilon \xi + z_m) \cdot \nu_m(\epsilon \xi + z_m) = \epsilon \frac{\partial U^m}{\partial \nu_m}(\epsilon \xi + z_m). \)

\[
\frac{\partial U^m}{\partial \nu_m}(\xi) := \lambda (\text{div}_E U^m(\xi)) N_m(\xi) + \mu (\nabla E U^m(\xi) + \nabla E U^m(\xi)^\top) N_m(\xi)
:= \epsilon \left[ \lambda (\text{div} U^m(\epsilon \xi + z_m)) N_m(\epsilon \xi + z_m) + \mu (\nabla U^m(\epsilon \xi + z_m) + \nabla U^m(\epsilon \xi + z_m)^\top) N_m(\epsilon \xi + z_m) \right]
= \frac{\partial U^m}{\partial \nu_m}(\epsilon \xi + z_m).
\]

Hence,

\[
\left\| \frac{\partial U^m}{\partial \nu_m}(\xi) \right\|_{L^2(\partial B_m)}^2 = \int_{\partial B_m} \left\| \frac{\partial U^m(\eta)}{\partial \nu_m(\eta)} \right\|^2 d\eta
= \int_{\partial D_m} \epsilon^2 \left\| \frac{\partial U^m(s)}{\partial \nu_m(s)} \right\|^2 \epsilon^{-2} ds, \quad [s := \epsilon \eta + z_m]
= \left\| \frac{\partial U^m}{\partial \nu_m} \right\|^2_{L^2(\partial D_m)},
\]

which gives us

\[
\left\| \frac{\partial U^m}{\partial \nu_m} \right\|_{L^2(\partial D_m)} \leq \left\| \frac{\partial U^m}{\partial \nu_m} \right\|_{L^2(\partial B_m)}. \tag{2.59}
\]

For every function \( \zeta_m \in H^1(\partial D_m) \), the corresponding \( U^\zeta_m \) exists in \( D_m \) as mentioned in (2.54) and then the corresponding functions \( U^m \) in \( B_m \) and the inequality (2.59) will be satisfied by these functions. Let \( A_{D_m} : H^1(\partial D_m) \to L^2(\partial D_m) \) and \( A_{B_m} : H^1(\partial B_m) \to L^2(\partial B_m) \) be the Dirichlet to Neumann maps.

Then we get the following estimate from (2.59).

\[
\left\| A_{D_m} \right\|_{L(H^1(\partial D_m), L^2(\partial D_m))} \leq \frac{1}{\epsilon} \left\| A_{B_m} \right\|_{L(H^1(\partial B_m), L^2(\partial B_m))}
\]

This implies that,

\[
\left\| \frac{\partial U^m}{\partial \nu_m} \right\|_{H^{-1}(\partial D_m)} \leq \left\| A_{D_m} \right\|_{L(H^1(\partial D_m), H^{-1}(\partial D_m))}
= \left\| A_{D_m} \right\|_{L(H^1(\partial D_m), L^2(\partial D_m))}
= \frac{1}{\epsilon} \left\| A_{B_m} \right\|_{L(H^1(\partial B_m), L^2(\partial B_m))}. \tag{2.60}
\]

Now, by (2.52) and (2.54),

\[
\left\| \frac{\partial U^m}{\partial \nu_m} \right\|_{H^{-1}(\partial D_m)} \leq \tilde{C}(\omega) \left\| A_{B_m} \right\|_{L(H^1(\partial B_m), L^2(\partial B_m))}. \tag{2.61}
\]

Hence the result is true as \( \left\| A_{B_m} \right\|_{L(H^1(\partial B_m), L^2(\partial B_m))} \) is bounded by a constant depending only on \( B_m \) through its size and Lipschitz character of \( B_m \).

\[ \square \]

**Definition 2.9.** We call \( \sigma_m \in L^2(\partial D_m) \) satisfying (2.5), the solution of the problem (1.1-1.3), as surface charge distributions. Using these surface charge distributions we define the total charge on each surface \( \partial D_m \) denoted by \( Q_m \) as

\[
Q_m := \int_{\partial D_m} \frac{\partial U^m}{\partial \nu_m}(s) ds. \tag{2.62}
\]
2.4.2. Estimates on the total charge $Q_m$, $m = 1, \ldots, M$. In the following proposition, we provide an approximate of the far-fields in terms of the total charges $Q_m$.

**Proposition 2.10.** The $P$-part, $U_p^\infty(\hat{x}, \hat{\theta})$, and the $S$-part, $U_s^\infty(\hat{x}, \hat{\theta})$, of the far-field pattern of the problem (1.1-1.3) have the following asymptotic expansions respectively:

\[
U_p^\infty(\hat{x}, \hat{\theta}) = \frac{1}{4\pi} \frac{1}{c_p^2} (\hat{x} \otimes \hat{x}) \sum_{m=1}^{M} \left[ e^{-i \frac{\hat{x} \cdot \hat{z}}{\rho} z_m} Q_m + O(a^2) \right],
\]

(2.63)

\[
U_s^\infty(\hat{x}, \hat{\theta}) = \frac{1}{4\pi} \frac{1}{c_s^2} (I - \hat{x} \otimes \hat{x}) \sum_{m=1}^{M} \left[ e^{-i \frac{\hat{x} \cdot \hat{z}}{\rho} z_m} Q_m + O(a^2) \right].
\]

(2.64)

if $\kappa_p a < 1$ and $\kappa_s a < 1$ where $O(a^2) \leq \hat{C}_a \omega a^2$ with

\[
\hat{C}_a := \frac{(|\alpha| + |\beta|) \| B \|_{L^2(\partial B)} \| \mathcal{A}_{B_m} \|_{L^1(\partial B_m), L^2(\partial B_m))}}{\max_{1 \leq m \leq M} \text{diam}(B_m)} \frac{1}{\min\{c_s, c_p\}}.
\]

**Proof of Proposition 2.10.** From (2.56), we have

\[
U^s(x) = \sum_{m=1}^{M} \int_{\partial D_m} \Gamma^s(x, s) \frac{\partial U^{\sigma_m}(s)}{\partial n_m(s)} ds, \text{ for } x \in \mathbb{R}^3 \setminus \left( \bigcup_{m=1}^{M} D_m \right).
\]

Substitution of the asymptotic behavior of the Kupradze tensor at infinity given in (2.2) in the above scattered field and comparing with (1.4), will allow us to write the $P$-part, $U_p^\infty(\hat{x}, \hat{\theta})$, and the $S$-part, $U_s^\infty(\hat{x}, \hat{\theta})$, of the far-field pattern of the problem (1.1-1.3) respectively as:

\[
U_p^\infty(\hat{x}, \hat{\theta}) = \frac{1}{4\pi} \frac{1}{c_p^2} (\hat{x} \otimes \hat{x}) \sum_{m=1}^{M} \int_{S_m} e^{-i \kappa_p \hat{x} \cdot \hat{z}} s \frac{\partial U^{\sigma_m}(s)}{\partial n_m(s)} ds
\]

\[
= \frac{1}{4\pi} \frac{1}{c_p^2} (\hat{x} \otimes \hat{x}) \sum_{m=1}^{M} \left[ e^{-i \kappa_p \hat{x} \cdot \hat{z}} z_m Q_m + \int_{S_m} \left[ e^{-i \kappa_p \hat{x} \cdot \hat{z}} s - e^{-i \kappa_p \hat{x} \cdot \hat{z}} z_m \right] \frac{\partial U^{\sigma_m}(s)}{\partial n_m(s)} ds \right],
\]

(2.65)

\[
U_s^\infty(\hat{x}, \hat{\theta}) = \frac{1}{4\pi} \frac{1}{c_s^2} (I - \hat{x} \otimes \hat{x}) \sum_{m=1}^{M} \int_{S_m} e^{-i \kappa_s \hat{x} \cdot \hat{z}} s \frac{\partial U^{\sigma_m}(s)}{\partial n_m(s)} ds
\]

\[
= \frac{1}{4\pi} \frac{1}{c_s^2} (I - \hat{x} \otimes \hat{x}) \sum_{m=1}^{M} \left[ e^{-i \kappa_s \hat{x} \cdot \hat{z}} z_m Q_m + \int_{S_m} \left[ e^{-i \kappa_s \hat{x} \cdot \hat{z}} s - e^{-i \kappa_s \hat{x} \cdot \hat{z}} z_m \right] \frac{\partial U^{\sigma_m}(s)}{\partial n_m(s)} ds \right].
\]

(2.66)

For every $m = 1, 2, \ldots, M$, we have from Lemma 2.8;

\[
\left| \int_{\partial D_m} \frac{\partial U^{\sigma_m}(s)}{\partial n_m(s)} ds \right| \leq ||1||_{H^1(\partial D_m)} \left| \frac{\partial U^{\sigma_m}}{\partial n_m} \right|_{H^{-1}(\partial D_m)} \leq \epsilon \| B \|_{\frac{1}{2}} \left| \frac{\partial U^{\sigma_m}}{\partial n_m} \right|_{H^{-1}(\partial D_m)} \leq (2.67)
\]

with $
\hat{C} := \frac{\hat{C}(\omega) \| B \|_{\frac{1}{2}} \| \mathcal{A}_{B_m} \|_{L^1(\partial B_m), L^2(\partial B_m))}}{\max_{1 \leq m \leq M} \text{diam}(B_m)} \frac{1}{\min\{c_s, c_p\}}$. It gives us the following estimate for any $\kappa$, i.e. $\kappa = \kappa_p$ or $\kappa_s$:

\[
\left| \int_{\partial D_m} \left[ e^{-i \kappa \hat{x} \cdot \hat{z}} s - e^{-i \kappa \hat{x} \cdot \hat{z}} z_m \right] \frac{\partial U^{\sigma_m}(s)}{\partial n_m(s)} ds \right| \leq \int_{\partial D_m} \left| e^{-i \kappa \hat{x} \cdot \hat{z}} s - e^{-i \kappa \hat{x} \cdot \hat{z}} z_m \right| \left| \frac{\partial U^{\sigma_m}(s)}{\partial n_m(s)} \right| ds
\]
Now substitution of (2.70) in (2.65) and (2.71) in (2.66) gives the required results (2.63), (2.64) respectively.

From (2.69), it follows that

\[
\int_{\partial D_m} [\epsilon^{1-\kappa^2}: s - \epsilon^{1-\kappa^2}: z_m] \frac{\partial U^{\sigma_m}}{\partial \nu_m(s)} ds \leq \hat{C} \kappa a^2, \text{ for } a \leq \frac{1}{\kappa_{\max}}. \tag{2.69}
\]

which means

\[
\int_{\partial D_m} [\epsilon^{1-\kappa^2}: s - \epsilon^{1-\kappa^2}: z_m] \frac{\partial U^{\sigma_m}}{\partial \nu_m(s)} ds \leq \hat{C} \kappa a^2, \text{ for } a \leq \frac{1}{\kappa_{\max}}. \tag{2.69}
\]

From (2.69), it follows that

\[
\int_{S_m} \left[ \epsilon^{1-\kappa^2}: s - \epsilon^{1-\kappa^2}: z_m \right] \frac{\partial U^{\sigma_m}}{\partial \nu_m(s)} ds < \hat{C} \kappa a^2, \text{ if } \epsilon \leq \frac{\min\{c_s, c_p\}}{\omega_{\max} \max_m \text{diam}(B_m)} \tag{2.70}
\]

\[
\int_{S_m} \left[ \epsilon^{1-\kappa^2}: s - \epsilon^{1-\kappa^2}: z_m \right] \frac{\partial U^{\sigma_m}}{\partial \nu_m(s)} ds < \hat{C} \kappa a^2, \text{ if } \epsilon \leq \frac{\min\{c_s, c_p\}}{\omega_{\max} \max_m \text{diam}(B_m)} \tag{2.71}
\]

Now substitution of (2.70) in (2.65) and (2.71) in (2.66) gives the required results (2.63), (2.64) respectively.

\[\Box\]

**Lemma 2.11.** For \(m = 1, 2, \ldots, M\), the absolute value of the total charge \(Q_m\) on each surface \(\partial D_m\) is bounded by \(\epsilon\), i.e.

\[|Q_m| \leq \hat{\epsilon}, \tag{2.72}\]

where \(\hat{\epsilon} := (|\alpha| + |\beta|) |\partial B| \hat{C} |A_{B_m}| \|L^{H^1(\partial B_m), L^2(\partial B_m)}\|\) with \(\partial B\) and \(\hat{C}\) are defined in (2.40) and (2.50) respectively.

**Proof of Lemma 2.11.** The proof follows as below:

\[
|Q_m| = \left| \int_{\partial D_m} \frac{\partial U^{\sigma_m}}{\partial \nu_m(s)} ds \right| \\
\leq \|1\|_{H^1(\partial D_m)} \left\| \frac{\partial U^{\sigma_m}}{\partial \nu_m(s)} \right\|_{H^{-1}(\partial D_m)} \\
\leq \left(2.61\right) \|1\|_{L^2(\partial D_m)} \hat{\epsilon} \omega |A_{B_m}| \|L^{H^1(\partial B_m), L^2(\partial B_m)}\| \\
\leq \left(2.32\right) \epsilon |\partial B| (|\alpha| + |\beta|) \hat{C} |A_{B_m}| \|L^{H^1(\partial B_m), L^2(\partial B_m)}\|. \\
\]

For \(s_m \in \partial D_m\), using the Dirichlet boundary condition (1.2), we have

\[
0 = U^i(s_m) = U^i(s_m) + \sum_{j=1}^{M} \int_{\partial D_j} \Gamma^\omega(s_m, s) \frac{\partial U^{\sigma_j}}{\partial \nu_j(s)} ds \\
= U^i(s_m) + \sum_{j=1}^{M} \int_{\partial D_j} \Gamma^\omega(s_m, s) \frac{\partial U^{\sigma_j}}{\partial \nu_j(s)} ds + \int_{\partial D_m} \Gamma^\omega(s_m, s) \frac{\partial U^{\sigma_m}}{\partial \nu_m(s)} ds \\
\]

\[\Box\]
To estimate \( \int_{\partial D_j} [\Gamma^\omega(s_m, s) - \Gamma^\omega(s_m, z_j)] \frac{\partial U^{\omega_j}(s)}{\partial v_j(s)} \, ds \) for \( j \neq m \), we have from Taylor series that,

\[
\Gamma^\omega(s_m, s) - \Gamma^\omega(s_m, z_j) = (s - z_j) \cdot R(s, m, s) - R(s_m, s) = \int_0^1 \nabla_2 \Gamma^\omega(s, m - \alpha(s - z_j)) \, d\alpha.
\]  

(2.74)

- From the definition of \( \Gamma^\omega(x, y) \) and by using the calculations made in (2.45), for \( s \in D_j \), we obtain

\[
|R(s_m, s)| \leq \max_{y \in D_j} |\nabla_2 \Gamma^\omega(s_m, y)| < \frac{1}{4\pi} \left[ \frac{C_9}{d_{m_j}^2} + C_{10} \right]
\]

(2.75)

with

\[
C_9 := 3 \left( \frac{1}{c_q^2} + \frac{1}{c_p^2} \right)
\]

and

\[
C_{10} := \frac{2}{c_q^4} \left[ \frac{1}{8} + \frac{1 - \left( \frac{1}{2} \kappa_{iw} \text{diam}(\Omega) \right)^{N_\Omega}}{1 - \frac{1}{2} \kappa_{iw} \text{diam}(\Omega)} \right] + \frac{1}{4\pi} \left( \frac{1}{4} + \frac{1 - \left( \frac{1}{2} \kappa_{ip} \text{diam}(\Omega) \right)^{N_\Omega}}{1 - \frac{1}{2} \kappa_{ip} \text{diam}(\Omega)} \right) + \frac{1}{2N_\Omega^{-1}}.
\]

Indeed, for \( x \in D_m \) and \( s \in D_j \), we have from (2.4);

\[
|\nabla_2 \Gamma^\omega(x, s)| \leq \frac{1}{4\pi} \frac{1}{\omega^2} \left[ \frac{3}{8} (\kappa_{iw}^2 + \kappa_{ip}^2) |x - s|^{-2} + \frac{1}{8} (6\kappa_{iw}^4 + 4\kappa_{ip}^4) \right]
\]

\[
+ \frac{1}{4\pi} \sum_{l=2}^{\infty} \frac{1}{(l - 2)!l(l + 2)} \frac{1}{\omega^2} \left( 2\kappa_{iw}^{l+2} + \kappa_{ip}^{l+2} \right) |x - s|^{-2}
\]

\[
\leq \frac{1}{4\pi} \frac{1}{\omega^2} \left[ \frac{3}{d_{m_j}^2} \left( \kappa_{iw}^2 + \kappa_{ip}^2 \right) + \frac{1}{4} \left( \kappa_{iw}^4 + \kappa_{ip}^4 \right) \right]
\]

\[
+ \sum_{l=2}^{\infty} \frac{1}{(l - 2)!l(l + 2)} \left( 2\kappa_{iw}^{l+2} + \kappa_{ip}^{l+2} \right) \text{diam}(\Omega)^{l-2}
\]

\[
\leq \frac{1}{4\pi} \left[ \frac{3}{d_{m_j}^2} \left( \frac{1}{c_q^2} + \frac{1}{c_p^2} \right) + \frac{1}{4} \left( \frac{\omega^2}{c_q^2} + \frac{\omega^2}{c_p^2} \right) \right]
\]

\[
+ \sum_{l=2}^{\infty} \frac{1}{(l - 2)!l(l + 2)} \left( \frac{\omega^2}{c_q^2} + \frac{\omega^2}{c_p^2} \right) \text{diam}(\Omega)^{l-2}
\]

[By recalling \( N_\Omega = [2\text{diam}(\Omega) \max\{\kappa_{iw}, \kappa_{ip}\} c^2] \) and using Lemma 2.6]

\[
\leq \frac{1}{4\pi} \left[ \frac{3}{d_{m_j}^2} \left( \frac{1}{c_q^2} + \frac{1}{c_p^2} \right) + \frac{1}{4} \left( \frac{\omega^2}{c_q^2} + \frac{\omega^2}{c_p^2} \right) \right]
\]

\[
+ \frac{\omega^2}{c_q^2} \sum_{l=0}^{N_\Omega-1} \left( \frac{1}{2} \kappa_{iw} \text{diam}(\Omega) \right)^l + \sum_{l=N_\Omega}^{\infty} \left( \frac{1}{2} \right)^l
\]

\[
+ \frac{\omega^2}{c_p^2} \sum_{l=0}^{N_\Omega-1} \left( \frac{1}{2} \kappa_{ip} \text{diam}(\Omega) \right)^l + \sum_{l=N_\Omega}^{\infty} \left( \frac{1}{2} \right)^l
\]
\[
= \frac{1}{4\pi} \left[ \frac{C_0}{d_{mj}^2} + C_{10} \right].
\] (2.76)

For \( m, j = 1, \ldots, M \), and \( j \neq m \), by making use of (2.75) and (2.67) we obtain the below:

\[
\left| \int_{\partial D_j} \left[ \Gamma^\omega(s_m, s) - \Gamma^\omega(s_m, z_j) \right] \frac{\partial U^{\sigma_j}(s)}{\partial v_j(s)} \, ds \right| = \left| \int_{\partial D_j} (s - z_j) \cdot R(s_m, s) \frac{\partial U^{\sigma_j}(s)}{\partial v_j(s)} \, ds \right|
\leq \int_{\partial D_j} |s - z_j| |R(s_m, s)| \left| \frac{\partial U^{\sigma_j}(s)}{\partial v_j(s)} \right| \, ds
< \frac{a}{4\pi} \left[ C_0 d^2 + C_{10} \right] \int_{\partial D_j} \left| \frac{\partial U^{\sigma_j}(s)}{\partial v_j(s)} \right| \, ds
< C \frac{a}{4\pi} \left[ C_0 d^2 + C_{10} \right] a.
\] (2.77)

Then (2.73) can be written as

\[
\int_{\partial D_m} \Gamma^0(s_m, s) \frac{\partial U^{\sigma_m}(s)}{\partial v_m(s)} \, ds + \int_{\partial D_m} \left[ \Gamma^\omega(s_m, s) - \Gamma^0(s_m, s) \right] \frac{\partial U^{\sigma_m}(s)}{\partial v_m(s)} \, ds
= -U^i(s_m) - \sum_{j=1, j \neq m}^{M} \Gamma^\omega(s_m, z_j)Q_j + O \left( (M - 1) \frac{a^2}{\varepsilon^2} \right).
\] (2.78)

By using the Taylor series expansions of the exponential term \( e^{i|s_m - s|} \), the above can also be written as,

\[
\int_{\partial D_m} \Gamma^0(s_m, s) \frac{\partial U^{\sigma_m}(s)}{\partial v_m(s)} \, ds + O(a) = -U^i(s_m) - \sum_{j=1, j \neq m}^{M} \Gamma^\omega(s_m, z_j)Q_j + O \left( (M - 1) \frac{a^2}{\varepsilon^2} \right).
\] (2.79)

Indeed,

\( \omega \leq \omega_{\text{max}} \) and for \( m = 1, \ldots, M \), we have

\[
\left| \int_{\partial D_m} \left[ \Gamma^\omega(s_m, s) - \Gamma^0(s_m, s) \right] \frac{\partial U^{\sigma_m}(s)}{\partial v_m(s)} \, ds \right|
\leq \int_{\partial D_m} \left| \Gamma^\omega(s_m, s) - \Gamma^0(s_m, s) \right| \left| \frac{\partial U^{\sigma_m}(s)}{\partial v_m(s)} \right| \, ds
\leq \int_{\partial D_m} \frac{\omega}{4\pi} \left[ \frac{2}{c_s^2} \sum_{l=0}^{\infty} \left( \frac{1}{2} \right)^l \kappa_{s l}^\omega \text{diam}(D_m)^l + \frac{1}{c_p^2} \sum_{l=0}^{\infty} \left( \frac{1}{2} \right)^l \kappa_{p l}^\omega \text{diam}(D_m)^l \right] \left| \frac{\partial U^{\sigma_m}(s)}{\partial v_m(s)} \right| \, ds
\leq \frac{\omega}{\pi} \left[ \frac{2}{c_s^2} \sum_{l=0}^{\infty} \left( \frac{1}{2} \right)^l \kappa_{s l}^\omega a^l + \frac{1}{c_p^2} \sum_{l=0}^{\infty} \left( \frac{1}{2} \right)^l \kappa_{p l}^\omega a^l \right] \cdot C a
\]
\leq \frac{C}{\pi} \left[ \frac{2}{c_s^2} + \frac{1}{c_p^2} \right] \omega a, \text{ for } \epsilon \leq \frac{\min\{c_s, c_p\}}{\omega_{\text{max}} \text{max}_m \text{diam}(B_m)}.
\]

Define \( U_m := \int_{\partial D_m} \Gamma^0(s_m, s) \frac{\partial U^{\sigma_m}(s)}{\partial v_m(s)} \, ds, s_m \in \partial D_m \). Then (2.79) can be written as

\[
U_m = -U^i(s_m) - \sum_{j=1, j \neq m}^{M} \Gamma^\omega(s_m, z_j)Q_j + O(a) + O \left( (M - 1) \frac{a^2}{\varepsilon^2} \right).
\] (2.80)

For \( m = 1, \ldots, M \), let \( \partial_m \in L^2(\partial D_m) \) be the surface charge distributions which define,
• The constant potentials \( \tilde{U}_m \in \mathbb{C}^{3 \times 1} \) as
\[
\int_{\partial D_m} \Gamma^0(s_m, s) \frac{\partial U^\sigma_m(s)}{\partial \nu_m(s)} \, ds = \tilde{U}_m := -U^i(z_m) - \sum_{j=1 \atop j \neq m}^M \Gamma^i(z_m, z_j) Q_j.
\] (2.81)

• The total charge \( Q_m \in \mathbb{C}^{3 \times 1} \) on the surface \( \partial D_m \) as
\[
Q_m := \int_{\partial D_m} \frac{\partial U^\sigma_m(s)}{\partial \nu_m(s)} \, ds.
\]

For \( m = 1, \ldots, M \), and \( l = 1, 2, 3 \), let \( \tilde{\sigma}_m^l \in L^2(\partial D_m) \) be the surface charge distributions which define,

• The constant potentials \( \tilde{U}_m^l \in \mathbb{C}^{3 \times 1} \) as
\[
\int_{\partial D_m} \Gamma^0(s_m, s) \frac{\partial U^\sigma_m^l(s)}{\partial \nu_m(s)} \, ds = \tilde{U}_m^l := - \left( U^i(z_m) \right)^l e_1 - \sum_{j=1 \atop j \neq m}^M \Gamma^l(z_m, z_j) Q_j(l) e_l, s_m \in \partial D_m
\]
with \( e_1 = (1, 0, 0) \top, e_2 = (0, 1, 0) \top \) and \( e_3 = (0, 0, 1) \top \).

• The charge \( Q_m^l \in \mathbb{C}^{3 \times 1} \) on surface \( S_m \) as
\[
Q_m^l := \int_{\partial D_m} \frac{\partial U^\sigma_m^l(s)}{\partial \nu_m(s)} \, ds,
\]
from which we can notice that \( \tilde{U}_m = \sum_{l=1}^3 \tilde{U}_m^l, \tilde{\sigma}_m = \sum_{l=1}^3 \tilde{\sigma}_m^l \) and \( Q_m = \sum_{l=1}^3 Q_m^l \).

Now, we set the electrical capacitance \( C_m \in \mathbb{C}^{3 \times 3} \) for \( 1 \leq m \leq M \) through
\[
Q_m^l = C_m \tilde{U}_m^l, l = 1, 2, 3 \quad \text{and hence} \quad Q_m = C_m \tilde{U}_m.
\] (2.83)

We can address the above also as, \([Q_1^1, Q_2^2, Q_3^3] = C_m [\tilde{U}_m^1, \tilde{U}_m^2, \tilde{U}_m^3] \), for each \( m = 1, 2, \ldots, M \).

**Lemma 2.12.** We have the following estimates for \( 1 \leq m \leq M \);
\[
\left\| \frac{\partial U^\sigma_m}{\partial \nu_m} - \frac{\partial U^\sigma_m}{\partial \nu_m} \right\|_{H^{-1}(\partial D_m)} = O \left( a + (M - 1) \frac{a^2}{d^2} \right),
\] (2.84)
\[
Q_m - \tilde{Q}_m = O \left( a^2 + (M - 1) \frac{a^3}{d^2} \right).
\] (2.85)

where the constants appearing in \( O(\cdot) \) depend only on the Lipschitz character of \( B_m \).

**Proof of Lemma 2.12.** By taking the difference between (2.80) and (2.81), we obtain
\[
U_m - \tilde{U}_m = \int_{\partial D_m} \Gamma^0(s_m, s) \left( \frac{\partial U^\sigma_m}{\partial \nu_m} - \frac{\partial U^\sigma_m}{\partial \nu_m} \right)(s) \, ds
\]
\[
= O(a) + O \left( (M - 1) \frac{a^2}{d^2} \right), \quad s_m \in \partial D_m.
\] (2.86)

Indeed, by using Taylor series,

- \( U^i(s_m) - U^i(z_m) = O(a) \).
- \( \Gamma^i(s_m, z_j) - \Gamma^i(z_m, z_j) = O \left( \frac{a}{d^2} \right) \) and the asymptoticity of \( Q_j \).

\footnote{Invertibility of D.L.P can be used.}
In operator form we can write (2.86) as,
\[ S_{mm}^* \left( \frac{\partial U_\sigma^m}{\partial v_m} - \frac{\partial U_\sigma^m}{\partial v_m} \right) (s_m) := \int_{\partial D_m} \Gamma^0(s_m, s) \left( \frac{\partial U_\sigma^m}{\partial v_m} - \frac{\partial U_\sigma^m}{\partial v_m} \right) (s) ds \]
\[ = O(a) + O \left( (M - 1) \frac{a^2}{d^2} \right), \quad s_m \in \partial D_m. \]
Here, \( S_{mm}^* : H^{-1}(\partial D_m) \to L^2(\partial D_m) \) is the adjoint of \( S_{mm} : L^2(\partial D_m) \to H^1(\partial D_m) \). We know that,
\[ \| S_{mm}^* \|_{L(H^{-1}(\partial D_m), L^2(\partial D_m))} = \| S_{mm} \|_{L(L^2(\partial D_m), H^1(\partial D_m))} \]
and
\[ \| S_{mm}^{-1} \|_{L(L^2(\partial D_m), H^{-1}(\partial D_m))} = \| S_{mm}^{-1} \|_{L(H^1(\partial D_m), L^2(\partial D_m))}, \]
then from (5.5) of Lemma 5.3, we obtain \( \| S_{mm}^{-1} \|_{L(L^2(\partial D_m), H^{-1}(\partial D_m))} = O(a^{-1}) \). Hence, we get the required results in the following manner.

- First,
\[ \left\| \frac{\partial U_\sigma^m}{\partial v_m} - \frac{\partial U_\sigma^m}{\partial v_m} \right\|_{H^{-1}(\partial D_m)} \leq \left\| S_{mm}^{-1} \right\|_{L(L^2(\partial D_m), H^{-1}(\partial D_m))} \left\| O(a) + O \left( (M - 1) \frac{a^2}{d^2} \right) \right\|_{L^2(\partial D_m)} \]
\[ = O \left( a + (M - 1) \frac{a^2}{d^2} \right). \]

- Second,
\[ |Q_m - \bar{Q}_m| = \left| \int_{\partial D_m} \left( \frac{\partial U_\sigma^m}{\partial v_m} - \frac{\partial U_\sigma^m}{\partial v_m} \right) (s) ds \right| \]
\[ \leq \left\| \frac{\partial U_\sigma^m}{\partial v_m} - \frac{\partial U_\sigma^m}{\partial v_m} \right\|_{H^{-1}(\partial D_m)} \left\| 1 \right\|_{H^1(\partial D_m)} \]
\[ = O \left( a^3 + (M - 1) \frac{a^4}{d^2} \right). \]

**Lemma 2.13.** For every \( 1 \leq m \leq M \), the capacitance \( \bar{C}_m \) and charge \( \bar{Q}_m \) are of the form;
\[ \bar{C}_m = \frac{\bar{C}_{B_m}}{\max_{1 \leq m \leq M} \text{diam}(B_m)} a \quad \text{and} \quad \bar{Q}_m = \frac{\bar{Q}_{B_m}}{\max_{1 \leq m \leq M} \text{diam}(B_m)} a, \quad (2.87) \]
where \( \bar{C}_{B_m} \) and \( \bar{Q}_{B_m} \) are the capacitance and the charge of \( B_m \) respectively.

**Proof of Lemma 2.13.** Take \( 0 < \epsilon \leq 1 \), \( z \in \mathbb{R}^3 \) and write, \( D_{\epsilon} := \epsilon B + z \subset \mathbb{R}^3 \). For \( \psi \in L^2(\partial D_{\epsilon}) \) and \( \psi \in L^2(\partial B) \), define the operators \( S_{1,2} : L^2(\partial D_{\epsilon}) \to H^1(\partial D_{\epsilon}) \) and \( S_{1,2}^* : L^2(\partial B) \to H^1(\partial B) \) as:
\[ S_{1,2} \psi \varepsilon(x) := \int_{\partial D_{\epsilon}} \Gamma^0(x, y) \psi(y) dy, \quad \text{and} \quad S_{1,2}^* \psi \varepsilon(x) := \int_{\partial B} \Gamma^0(x, \eta) \psi(\eta) d\eta. \]
Define \( U_{\psi \varepsilon} \) and \( U_{\psi} \) as the functions on \( \bar{D}_{\epsilon} \) and \( \bar{B} \) respectively in the similar way of (2.54). Then the operators
\[ S_{1,2} U_{\psi \varepsilon}(x) := \int_{\partial D_{\epsilon}} \Gamma^0(x, y) \frac{\partial U_{\psi \varepsilon}}{\partial v_y}(y) dy, \quad \text{and} \quad S_{1,2}^* U_{\psi \varepsilon}(x) := \int_{\partial B} \Gamma^0(x, \eta) \frac{\partial U_{\psi \varepsilon}}{\partial v_{\eta}}(\eta) d\eta. \]
define the corresponding potentials $\bar{U}, \bar{U}_B$ on the surfaces $\partial D_e$ and $\partial B$ w.r.t the surface charge distributions $\psi_e$ and $\psi$ respectively. Let, these potentials be equal to some constant vector $D \in \mathbb{C}^{3 \times 1}$. Let the total charge of these conductors $D_e$ be $Q_e$ and $Q_B$, and the capacitances are $C_e$ and $C_B$ respectively. Then we can write these as,

$$\bar{U}_e := \mathcal{S}^i U^\psi_e (x) = D, \quad \bar{U}_B := \mathcal{S}^i_U U^\psi (\xi) = D, \forall x \in \partial D_e, \forall \xi \in \partial B.$$ 

We have by definitions, $Q_e = \int_{\partial D_e} \frac{\partial U^\psi_e}{\partial \nu_y} (y) dy, \ Q_B = \int_{\partial B} \frac{\partial U^\psi}{\partial \nu_y} (\eta) d\eta$, and $C_e \bar{U}_e = Q_e, \ C_B \bar{U}_B = Q_B$. Observe that,

$$\begin{align*}
D = \mathcal{S}^i U^\psi_e(x) &= \int_{\partial D_e} \Gamma^0(x,y) \frac{\partial U^\psi_e}{\partial \nu_y}(y) dy \\
&= \int_{\partial B} \frac{1}{\epsilon} \Gamma^0(\xi,\eta) \frac{1}{\epsilon} \frac{\partial U^\psi_e}{\partial \nu_y}(\epsilon \eta + z)^2 d\eta \\
&= \int_{\partial D_e} \Gamma^0(\xi,\eta) \frac{\partial U^\psi_e}{\partial \nu_y}(\eta) d\eta \\
&= \mathcal{S}_U^i U^\psi_e(\xi). \quad \left[\hat{\psi}_e(\eta) := \psi_e(\epsilon \eta + z)\right]
\end{align*}$$

Hence, $U^\psi_e = \bar{U}$ and $U^\psi = \bar{U}$. Now we have,

$$\begin{align*}
Q_e &= \int_{\partial D_e} \frac{\partial U^\psi_e}{\partial \nu_y} (y) dy \int_{\partial D_e} \frac{\partial U^\psi}{\partial \nu_y} (y) dy, \\
&= \int_{\partial B} \frac{1}{\epsilon} \frac{\partial U^\psi}{\partial \nu_y}(\epsilon \eta + z)^2 d\eta = \epsilon \int_{\partial B} \frac{\partial U^\psi}{\partial \nu_y}(\epsilon \eta + z) d\eta, \\
&= \epsilon \int_{\partial B} \frac{\partial U^\psi}{\partial \nu_y}(\eta) d\eta = \epsilon \int_{\partial B} \frac{\partial U^\psi}{\partial \nu_y}(\eta) d\eta, \\
&= \epsilon Q_B
\end{align*}$$

which gives us,

$$C_e D = C_e \bar{U}_e = Q_e = \epsilon Q_B = \epsilon C_B \bar{U}_B = \epsilon C_B D.$$ 

It is true for every constant vector $D$ and hence $C_e = \epsilon C_B$. As we have $D_m = \epsilon B_m + z_m$ and $a = \max_{1 \leq m \leq M} diam(D_m) = \epsilon \max_{1 \leq m \leq M} diam(B_m)$, we obtain

$$Q_m = \epsilon Q_B = \frac{Q_B}{\max_{1 \leq m \leq M} diam(B_m)} a \quad \text{and} \quad C_m = \epsilon C_B = \frac{C_B}{\max_{1 \leq m \leq M} diam(B_m)} a.$$ 

\[ \Box \]

**Lemma 2.14.** For $m = 1, 2, \ldots, M$, the elastic capacitances $C_m \in \mathbb{C}^{3 \times 3}$ defined through (2.83) are non-singular.

**Proof of Proposition 2.14.** As the capacitances $C_m$ depend only on the scatterers, let $\sigma^l_m \in L^2(\partial D_m)$ be surface charge distributions which define the potentials $e_l$ for $l = 1, 2, 3$. i.e.

$$\int_{\partial D_m} \Gamma^0(s_m,s) \frac{\partial U^\sigma^l_m}{\partial \nu_m} (s) ds = e_l =: U^l_m, \ \text{for} \ l = 1, 2, 3, \ m = 1, \ldots, M.$$ (2.88)
We also have
\[ \int_{\partial D_m} \frac{\partial U^{\sigma_m \lambda}}{\partial n_m}(s) \, ds, \int_{\partial D_m} \frac{\partial U^{\sigma_m \lambda}}{\partial n_m^2}(s) \, ds, \int_{\partial D_m} \frac{\partial U^{\sigma_m \lambda}}{\partial n_m^3}(s) \, ds = C_m \left[ U_m^1, U_m^2, U_m^3 \right] = C_m. \]
Hence, it is enough if we show that the matrix 
\[ \left[ \int_{\partial D_m} \left( \frac{\partial U^{\sigma_m \lambda}}{\partial n_m} \right)_l(s) \, ds \right]_{j=1}^3 \]
is invertible. In order to prove this, assume the linear combination
\[ \sum_{l=1}^3 a_l \int_{\partial D_m} \frac{\partial U^{\sigma_m \lambda}}{\partial n_m}(s) \, ds = 0 \]
for the scalars \( a_l \in \mathbb{C} \). From (2.88), we can deduce that
\[ \int_{\partial D_m} \int_{\partial D_m} \Gamma^0(s_{m_1}, s_{m_2}) \left( \sum_{l=1}^3 a_l \frac{\partial U^{\sigma_m \lambda}}{\partial n_m}(s_{m_2}) \right) \frac{\partial U^{\sigma_m \lambda}}{\partial n_m}(s_{m_1}) \, ds_{m_1} \, ds_{m_2} = 0, \quad j = 1, 2, 3, \]
and hence
\[ \int_{\partial D_m} \int_{\partial D_m} \Gamma^0(s_{m_1}, s_{m_2}) \left( \sum_{l=1}^3 a_l \frac{\partial U^{\sigma_m \lambda}}{\partial n_m}(s_{m_2}) \right) \left( \sum_{l=1}^3 a_l \frac{\partial U^{\sigma_m \lambda}}{\partial n_m}(s_{m_1}) \right) \, ds_{m_1} \, ds_{m_2} = 0. \]
The positivity of the single layer operator implies, \( \sum_{l=1}^3 a_l \frac{\partial U^{\sigma_m \lambda}}{\partial n_m}(s) = 0, \quad s \in \partial D_m. \)
Again now by making use of (2.88), we deduce
\[ \sum_{l=1}^3 a_l \epsilon_l = \int_{\partial D_m} \Gamma^0(s, s) \left( \sum_{l=1}^3 a_l \frac{\partial U^{\sigma_m \lambda}}{\partial n_m}(s) \right) \, ds = 0, \quad s \in \partial D_m, \]
and hence \( a_l = 0 \) for \( l = 1, 2, 3. \)

**Proposition 2.15.** For \( m = 1, 2, \ldots, M \), the total charge \( Q_m \) on each surface \( \partial D_m \) of the small scatterer \( D_m \) can be calculated from the algebraic system
\[ C_m^{-1} \hat{Q}_m = -U^i(z_m) - \sum_{j=1}^M \Gamma^\omega(z_m, z_j) \hat{C}_j^{-1} \hat{Q}_j, \quad (2.89) \]
with an error of order \( O \left( (M-1) \frac{d^2}{r} + (M-1)^2 \frac{d^3}{r^2} \right) \).

**Proof of Proposition 2.15.** We can rewrite (2.81) as
\[ C_m^{-1} \hat{Q}_m = -U^i(z_m) - \sum_{j=1}^M \Gamma^\omega(z_m, z_j) Q_j \]
\[ = -U^i(z_m) - \sum_{j=1}^M \Gamma^\omega(z_m, z_j) \hat{Q}_j - \sum_{j=1}^M \Gamma^\omega(z_m, z_j) (Q_j - \hat{Q}_j) \]
\[ = -U^i(z_m) - \sum_{j=1}^M \Gamma^\omega(z_m, z_j) \hat{Q}_j + O \left( (M-1) \frac{d^2}{r} + (M-1)^2 \frac{d^3}{r^2} \right), \]
where we used (2.85) and the fact \( \Gamma^\omega(z_m, z_j) = O \left( \frac{1}{d^2} + \omega \right), \quad \omega \leq \omega_{\max} \) and \( d \leq d_{\max}. \) Indeed,
\[ |\Gamma^\omega(z_m, z_j)| \leq \frac{1}{4\pi} \frac{1}{\omega^2} \left( \kappa_1^{\omega} + \kappa_2^{\omega} \right) |z_m - z_j|^{-1} + \frac{1}{4\pi} \sum_{l=1}^\infty \frac{1}{(l-1)! (l+2)} \frac{1}{\omega^{l+2}} \left( 2\kappa_1^{\omega} + \kappa_2^{\omega} \right) |z_m - z_j|^{-l-1}. \]
invertible. We discuss its invertibility in Section 4.

The above linear algebraic system is solvable for the 3D vectors \( \vec{m} \) and \( \vec{s} \).

\[ B = \begin{pmatrix} \frac{1}{d_1} \left( \frac{1}{c_s^2} \right) + \frac{1}{C_p} \sum_{i=1}^{N_1} \left( \frac{1}{2\kappa_{sw} \text{diam}(\Omega)} \right)^{l-1} + \frac{1}{2N_{l-1}} \right) + \frac{1}{2N_{l-1}} \right) \]

\[ = \frac{1}{4\pi} \left[ C_7 + C_8 \right] \]

with

\[ C_7 := \left[ \frac{1}{c_s^2} + \frac{2}{C_p^2} \right] \]

and

\[ C_8 := 2 \frac{\kappa_{sw}}{c_s^2} \left( \frac{1 - \left( \frac{1}{2\kappa_{sw} \text{diam}(\Omega)} \right)^{N_1}}{1 - \frac{1}{2N_{l-1}}} \right) + \frac{\kappa_{pw}}{C_p^2} \left( \frac{1 - \left( \frac{1}{2\kappa_{pw} \text{diam}(\Omega)} \right)^{N_1}}{1 - \frac{1}{2N_{l-1}}} \right) \]

2.4.3. The algebraic system. Define the algebraic system,

\[
\hat{C}_m^{-1} \hat{Q}_m := -U^i(z_m) - \sum_{j=1}^{M} \Gamma^w(z_m, z_j) \hat{C}_j (\hat{C}_j^{-1} \hat{Q}_j),
\]

for all \( m = 1, 2, \ldots, M \). It can be written in a compact form as

\[ B \hat{Q} = U^I, \]

where \( \hat{Q}, U^I \in \mathbb{C}^{3M \times 1} \) and \( B \in \mathbb{C}^{3M \times 3M} \) are defined as

\[ B := \begin{pmatrix} \hat{C}_1^{-1} & -\Gamma^w(z_1, z_2) & -\Gamma^w(z_1, z_3) & \cdots & -\Gamma^w(z_1, z_M) \\
-\Gamma^w(z_2, z_1) & -\hat{C}_2^{-1} & -\Gamma^w(z_2, z_3) & \cdots & -\Gamma^w(z_2, z_M) \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
-\Gamma^w(z_M, z_1) & -\Phi(z_M, z_2) & \cdots & -\Gamma^w(z_M, z_{M-1}) & -\hat{C}_M^{-1} \end{pmatrix}, \]

\[ \hat{Q} := \begin{pmatrix} \hat{Q}_1^T & \hat{Q}_2^T & \cdots & \hat{Q}_M^T \end{pmatrix}^T \]

and \( U^I := \begin{pmatrix} U^i(z_1)^T & U^i(z_2)^T & \cdots & U^i(z_M)^T \end{pmatrix}^T \). The above linear algebraic system is solvable for the 3D vectors \( \hat{Q}_j \), \( 1 \leq j \leq M \), when the matrix \( B \) is invertible. We discuss its invertibility in Section 4.
Hence, Theorem 1.2 is proved by setting
\[ \tilde{C}_m^{-1}(\tilde{Q}_m - \tilde{Q}_m) = -\sum_{j=1}^{M} \Gamma^m(z_m, z_j) \left( \tilde{Q}_j - \tilde{Q}_j \right) + O \left( (M-1) \frac{a^3}{d} + (M-1)^2 \frac{d^4}{d^3} \right). \] (2.93)
for \( m = 1, 2, \ldots, M \). Considering the above system of equations (2.93) in the place of (2.91) and then by making use of the Corollary 4.3 and the fact that acoustic capacitances of the scatterers are bounded above and below by their diameters multiplied by constants which depend only on the Lipschitz character of \( B_m \)'s, see [11, Lemma 2.11 and Remark 2.23], we obtain
\[ \sum_{m=1}^{M} (\tilde{Q}_m - \tilde{Q}_m) = O \left( M(M-1) \frac{a^3}{d} + M(M-1)^2 \frac{d^4}{d^3} \right). \] (2.94)

2.5. End of the proof of Theorem 1.2. The use of (2.85), (2.94) in (2.63) and (2.64) allows us to represent the asymptotic expansions of the \( P \) part, \( U_p^\infty(\hat{x}, \hat{\theta}) \), and the \( S \) part, \( U_s^\infty(\hat{x}, \hat{\theta}) \), of the far-field pattern of the problem (1.1-1.3) in terms of \( \tilde{Q}_m \) respectively as below;
\[ U_p^\infty(\hat{x}, \hat{\theta}) = \frac{1}{4\pi c_p} \left( \hat{x} \times \hat{\theta} \right) \sum_{m=1}^{M} e^{-i \frac{2\pi}{\tilde{Q}_m} \cdot \hat{x}} \left[ Q_m + O(a^2) \right] = \frac{1}{4\pi c_p} \left( \hat{x} \times \hat{\theta} \right) \sum_{m=1}^{M} e^{-i \frac{2\pi}{\tilde{Q}_m} \cdot \hat{x}} \left[ \tilde{Q}_m + (Q_m - \tilde{Q}_m) \right] + O(a^2) = \frac{1}{4\pi c_p} \left( \hat{x} \times \hat{\theta} \right) \sum_{m=1}^{M} e^{-i \frac{2\pi}{\tilde{Q}_m} \cdot \hat{x}} \tilde{Q}_m + O \left( a^2 + (M-1)^2 \frac{a^3}{d^2} + M(M-1) \frac{d^4}{d^3} \right) \] (2.95)
\[ U_s^\infty(\hat{x}, \hat{\theta}) = \frac{1}{4\pi c_s} \left( \hat{x} \times \hat{\theta} \right) \sum_{m=1}^{M} e^{-i \frac{2\pi}{\tilde{Q}_m} \cdot \hat{x}} \left[ Q_m + O(a^2) \right] = \frac{1}{4\pi c_s} \left( \hat{x} \times \hat{\theta} \right) \sum_{m=1}^{M} e^{-i \frac{2\pi}{\tilde{Q}_m} \cdot \hat{x}} \tilde{Q}_m + O \left( a^2 + (M-1)^2 \frac{a^3}{d^2} + M(M-1)^2 \frac{d^4}{d^3} \right) \] (2.96)

Hence, Theorem 1.2 is proved by setting \( \tilde{\sigma}_m := \frac{\tilde{Q}_m}{\tilde{B}_m} \) as the surface density which defines \( \tilde{Q}_m \). Finally, let us remark that

1. The constant \( \hat{c} := \left[ \frac{1}{4\pi} \left( \tilde{C}_T + \tilde{C}_S \tilde{d}_{m} \right) \right] \) appearing in Proposition 2.2 will serve our purpose in Theorem 1.2 by defining \( c_0 := \hat{c} \max_{1 \leq m \leq M} \text{diam}(B_m) \) respectively.
2. The coefficients \( \tilde{\sigma}_m \tilde{U}_m^{-1}, \tilde{Q}_m, \tilde{C}_m \) plays the roles of \( \sigma_m, Q_m, C_m \) respectively in Theorem 1.2.
3. The constant appearing in \( O \left( Ma^2 + M(M-1)^2 \frac{a^3}{d^2} + M(M-1)^2 \frac{d^4}{d^3} \right) \) is
\[ C^e \max \left\{ 1 + \frac{\max_{1 \leq m \leq M} \tilde{C}_B_m}{\text{diam}(B_m)}, \frac{C_T + \tilde{C}_S \tilde{d}_{m} \max_{1 \leq m \leq M} \text{diam}(B_m)}{4\pi}, 1 + \frac{\tilde{C} \omega}{C^e \min\{c_s, c_p\}} \right\} \]
with \( C^e := \max_{1 \leq m \leq M} \tilde{C}_m |\tilde{B}| \) max \( \left\{ \left( \frac{|a| + |b|}{\min\{c_s, c_p\}} \right) + \frac{\tilde{C}}{\frac{2}{c} \tilde{C} + \frac{1}{c}} \right\} \). The constants \( |\tilde{B}| \) and \( \tilde{C} \) are defined in Propositin 2.7 and Proposition 2.10 respectively and the constant \( C_{6m} \) is same as the constant \( C_0 \) in Lemma 5.3 but associated with the scatterer \( D_m \).
4. The constant \( a_0 \) appearing in (1.6) of Theorem 1.2 is the minimum among
\[
\frac{1}{\omega_{\text{max}}} \min \left\{ c_s, c_p, \sqrt{\frac{\pi}{2}} \frac{1}{\mu^2} \left[ \frac{1}{1 - 2j} \right] \right\},
\]
and
\[
\omega_{\text{max}} \max_{1 \leq m \leq M} \frac{\omega_{\text{max}}}{\omega_{\text{max}}} \left( \frac{4x^2 + 17a + 2}{2\mu} \right) \left( \frac{1}{j} + \frac{1}{j} \right) \leq \omega_{\text{max}} \left( \frac{1}{j} + \frac{1}{j} \right) \left( \frac{1}{j} + \frac{1}{j} \right).
\]

5. The constant \( c_1 \) appearing in (1.11) of Theorem 1.2 is
\[
\frac{5\pi^2}{3} \left( \frac{\lambda + 2\mu}{\max_{1 \leq m \leq M} \lambda^2(B_m)} \right) \left( \frac{\lambda + 2\mu}{\max_{1 \leq m \leq M} \mu^2(B_m)} \right) \left( \frac{\lambda + 2\mu}{\max_{1 \leq m \leq M} C^2(B_m)} \right) \left( \frac{\lambda + 2\mu}{\max_{1 \leq m \leq M} C^2(B_m)} \right)
\]
with \( \lambda^2(B_m) \) denoting the acoustic capacitance of the bodies \( B_m \) and it follows from Corollary 4.3 and from [11, Lemma 2.11].

From the last points, we see that the constants appearing in Theorem 1.2 depend only on \( d_{\text{max}}, \omega_{\text{max}} \) and \( B_m \)’s through their diameters, capacitances and the norms of the boundary operators \( S_{B_m}^{-1} \) : \( H^1(\partial B_m) \rightarrow L^2(\partial B_m), \left( \frac{1}{2}I + D_{B_m}^* \right)^{-1}, L^2(\partial B_m) \rightarrow L^2(\partial B_m) \) and \( A_{B_m} : H^1(\partial B_m) \rightarrow L^2(\partial B_m) \). As it was explained in the acoustic case in [11, Remark 2.25], the capacitances and the bounds of the operators \( S_{B_m}^{-1} \) and \( \left( \frac{1}{2}I + D_{B_m}^* \right)^{-1} \) depend on \( B_m \)’s actually only through their Lipschitz character.

---

2.6. Proof of remark 1.3. It can be proved in the similar way of see [11, Remark 1.4]. For \( m = 1, \ldots, M \), fixed, we distinguish between the obstacles \( D_j, j \neq m \) which are near to \( D_m \) from the ones which are far from \( D_m \) as follows. Let \( \Omega_m, 1 \leq m \leq M \) be the balls of center \( z_m \) and of radius \( (\frac{3}{2} + \alpha) \) with \( 0 < \alpha \leq 1 \). The bodies lying in \( \Omega_m \) will fall into the category, \( N_m \), of near by obstacles and the others into the category, \( F_m \), of far obstacles to \( D_m \). Since the obstacles \( D_m \) are balls with same diameter, the number of obstacles near by \( D_m \) will not exceed \( \left( \frac{a + 2d}{a + d} \right)^3 \) \( \frac{\pi^2}{4\pi\pi} \left( \frac{a + 2d}{a + d} \right)^3 \).

With this observation, instead of (1.7-1.8), the P and the S parts of the far field will have the asymptotic expansions (1.15-1.16). Indeed,

• For the bodies \( D_j \in N_m, j \neq m \), we have the estimate (2.75) but for the bodies \( D_j \in F_m \), we obtain the following estimate

\[
|R(s_m, s)| \leq \max_{y \in D_j} |\nabla_y \Gamma^\infty(s_m, y)| < \frac{1}{4\pi} \left[ \frac{C_0}{d_m} + C_{10} \right].
\]

• Due to the estimates (2.75) and (2.97), corresponding changes will take place in (2.77-2.79), (2.80), (2.84-2.85) and in (2.89-2.90) which inturn modify (2.93-2.94) and hence the asymptotic expansion (1.7) as follows

\[
U^\infty(\hat{x}, \hat{\theta}) = \frac{1}{4\pi c_p} (\hat{x} \otimes \hat{\theta}) \left[ \sum_{m=1}^{M} e^{-i\hat{x} \cdot \hat{\theta}} \cdot \frac{Q_m + O \left( M a^2 + M (M - 1) a d^3 + a d^2 \right)}{d^2} \right] + M (M - 1)^2 \frac{a^4}{d^2} + M (M - 1) \left( \frac{a + 2d^2}{a + d} \right)^3 \frac{a^4}{d^2} + M \left( \frac{a + 2d^2}{a + d} \right) d^2
\]

• Since \( \kappa \leq \kappa_{\text{max}}, d \leq d^0, 0 < \alpha \leq 1 \) and \( \frac{a}{d} < \infty \), we have

\[
\left( \frac{a + 2d^2}{a + d} \right) = d^{\alpha - 1} \frac{a d - \alpha + 2}{a d - \alpha + 1} = O(d^{\alpha - 1}),
\]
which can be used to derive (1.15) from (2.98). In the similar way, we can obtain (1.16). Finally, it is easily seen that the above analysis applies also for non-flat Lipschitz domains \( D_m \) by using the double inclusions (1.19) and the fact that \( t_m \)’s are uniformly bounded from below by a positive constant.
3. Accuracy of the Foldy-Lax approximation. We define the scalar P-part, \( U_p^\infty (\hat{x}, \theta) \), and the scalar S-part, \( U_s^\infty (\hat{x}, \theta) \), of the far-field pattern of the problem (1.1-1.3) respectively as:

\[
U_p^\infty (\hat{x}, \theta) := 4\pi e_p^2 (\hat{x} \cdot U_p^\infty (\hat{x}, \theta)) = \sum_{m=1}^{M} \hat{x} e^{-i \frac{\hat{x} \cdot z_m}{\epsilon}} \tilde{Q}_m + O \left( Ma^2 + M(M-1)\frac{a^4}{\epsilon^2} + M(M-1)^2 \frac{a^4}{\epsilon^2} \right),
\]

(3.1)

\[
U_s^\infty (\hat{x}, \theta) := 4\pi c_s^2 (\hat{x} \cdot U_s^\infty (\hat{x}, \theta)) = \sum_{m=1}^{M} \hat{x} e^{-i \frac{\hat{x} \cdot z_m}{\epsilon}} \tilde{Q}_m + O \left( Ma^2 + M(M-1)\frac{a^4}{\epsilon^2} + M(M-1)^2 \frac{a^4}{\epsilon^2} \right).
\]

(3.2)

From (3.1) and (3.2), we can write the scalar P and the scalar S parts of the far-field pattern as

\[
U_p^\infty (\hat{x}, \theta) = \sum_{m=1}^{M} \hat{x} e^{-i \frac{\hat{x} \cdot z_m}{\epsilon}} \tilde{Q}_m
\]

(3.3)

\[
U_s^\infty (\hat{x}, \theta) = \sum_{m=1}^{M} \hat{x} e^{-i \frac{\hat{x} \cdot z_m}{\epsilon}} \tilde{Q}_m
\]

(3.4)

with the error of order \( O \left( Ma^2 + M(M-1)\frac{a^4}{\epsilon^2} + M(M-1)^2 \frac{a^4}{\epsilon^2} \right) \) and \( \tilde{Q}_m \) can be obtained from the linear algebraic system (2.92). Let us denote the inverse of \( B \) by \( B \) and the corresponding \( 3 \times 3 \) blocks of \( B \) by \( B_{mj}, m, j = 1, \ldots, M \). Then we can rewrite (3.3) and (3.4), with the same error, as follows

\[
U_p^\infty (\hat{x}, \theta) = \sum_{m=1}^{M} \sum_{j=1}^{M} e^{-i \frac{\hat{x} \cdot z_m}{\epsilon}} \hat{x}^\top B_{mj} U^j(z_j, \theta)
\]

(3.5)

\[
U_s^\infty (\hat{x}, \theta) = \sum_{m=1}^{M} \sum_{j=1}^{M} e^{-i \frac{\hat{x} \cdot z_m}{\epsilon}} (\hat{x}^\top B_{mj} U^j(z_j, \theta)
\]

(3.6)

for a given incident direction \( \theta \) and observation direction \( \hat{x} \). From (3.5) and (3.6), we can get the scalar P and the scalar S parts of the far-field patterns corresponding to plane incident P-wave \( U_p^\infty (x, \theta) \) and S-wave \( U_s^\infty (x, \theta) \), denote respectively by \( U_p^\infty \cdot p (\hat{x}, \theta) \), \( U_s^\infty \cdot p (\hat{x}, \theta) \), \( U_p^\infty \cdot S (\hat{x}, \theta) \), \( U_s^\infty \cdot S (\hat{x}, \theta) \) as below:

\[
U_p^\infty \cdot p (\hat{x}, \theta) = \sum_{m=1}^{M} \sum_{j=1}^{M} e^{-i \frac{\hat{x} \cdot z_m}{\epsilon}} \hat{x}^\top B_{mj} \theta e^{i \frac{\hat{x} \cdot z_j}{\epsilon}}
\]

(3.7)

\[
U_p^\infty \cdot S (\hat{x}, \theta) = \sum_{m=1}^{M} \sum_{j=1}^{M} e^{-i \frac{\hat{x} \cdot z_m}{\epsilon}} (\hat{x}^\top B_{mj} \theta) e^{i \frac{\hat{x} \cdot z_j}{\epsilon}}
\]

(3.8)

\[
U_p^\infty \cdot s (\hat{x}, \theta) = \sum_{m=1}^{M} \sum_{j=1}^{M} e^{-i \frac{\hat{x} \cdot z_m}{\epsilon}} \hat{x}^\top B_{mj} \theta (\hat{x}) e^{i \frac{\hat{x} \cdot z_j}{\epsilon}}
\]

(3.9)

\[
U_s^\infty \cdot s (\hat{x}, \theta) = \sum_{m=1}^{M} \sum_{j=1}^{M} e^{-i \frac{\hat{x} \cdot z_m}{\epsilon}} (\hat{x}^\top B_{mj} \theta (\hat{x})) e^{i \frac{\hat{x} \cdot z_j}{\epsilon}}
\]

(3.10)

All the far-field patterns (3.7-3.10) are valid with the same error which is equal to the error in (3.3-3.4). We distinguish the following two cases in terms of the minimum distance \( d := d(\epsilon) \):
• Case 1: We have a finite $\omega$, $d \ll 1$ and $(M - 1)\frac{a^2}{\pi} \ll 1$. In this case, using the formulas (3.7-3.10) are valid with the error of order $o\left((M - 1)\frac{a^2}{\pi}\right)$ as $\epsilon \to 0$. Indeed,

$$O\left(Ma^2 + M(M - 1)\frac{a^3}{d^2} + M(M - 1)^2\frac{a^4}{d^3}\right)^6 = o\left((M - 1)\frac{a^2}{\pi}\right).$$

Now, remark that $\tilde{Q}_m := -C_m U^i(z_m) + C_m \sum_{j=1}^{M} \Gamma^\omega(z_m, z_j) C_j U^i(z_j) + O\left((M - 1)^2\frac{a^3}{\pi}\right)$, where $C_m U^i(z_m)$ behaves as $a$ and $C_m \sum_{j=m}^{M} \Phi(z_m, z_j) C_j U^i(z_j)$ behaves as $(M - 1)\frac{a^2}{\pi}$. This means that (3.7-3.10) reduces to

$$U_p^\infty(\hat{x}, \theta) = -\sum_{m=1}^{M} e^{i\frac{\pi}{\epsilon} (\theta - \hat{x}) z_m^\top C_m \theta} + \sum_{j=1}^{M} \sum_{j \neq m} e^{-i\frac{\pi}{\epsilon} \hat{x}^\top z_m C_m C_j \Gamma^\omega(z_m, z_j) \theta} e^{i\frac{\pi}{\epsilon} \hat{x}^\top z_j},$$

$$U_s^\infty(\hat{x}, \theta) = -\sum_{m=1}^{M} e^{i\frac{\pi}{\epsilon} (\theta - \hat{x}) z_m^\top C_m \theta} + \sum_{j=1}^{M} \sum_{j \neq m} e^{-i\frac{\pi}{\epsilon} \hat{x}^\top z_m C_m C_j \Gamma^\omega(z_m, z_j) \theta} e^{i\frac{\pi}{\epsilon} \hat{x}^\top z_j},$$

which are valid with an error of order $o\left((M - 1)\frac{a^2}{\pi}\right)$ and the first term in each of the above equations models the Born approximation and second term models the first order interaction between the scatterers. As a conclusion, when we use (3.7-3.10) we compute the field generated by the first interaction between the collection of the scatterers $z_m, m = 1, \ldots, M$.

• Case 2: We assume that there exists a positive constant $d_0$ such that $d_0 \leq d$. In this case, the formulas (3.7-3.10) are reduced to

$$U_p^\infty(\hat{x}, \theta) = -\sum_{m=1}^{M} e^{i\frac{\pi}{\epsilon} (\theta - \hat{x}) z_m^\top C_m \theta} + O(a^2),$$

$$U_s^\infty(\hat{x}, \theta) = -\sum_{m=1}^{M} e^{i\frac{\pi}{\epsilon} (\theta - \hat{x}) z_m^\top C_m \theta} + O(a^2),$$

as $\epsilon \to 0$. In this case, we have only the Born approximation.

An example of distribution of the scatterers in case 1 is $M = \epsilon^{-1}$ and $d(\epsilon) = \epsilon^1$. Note that in case 2 we have a lower bound on the distances between the scatterers. This explains why we are in the Born regime. Remark also that, in this case, $M$ is uniformly bounded since the obstacles are included in the bounded domain $\Omega$.

\footnote{Since $Ma^2 = (M - 1)\frac{a^2}{\pi} d, M(M - 1)\frac{a^3}{\pi^2} = (M - 1)\frac{a^3}{\pi^2} (M - 1)\frac{a^2}{\pi^2}$ and the assumptions $(M - 1)\frac{a}{\pi} \ll 1, d \ll 1$.}
4. Solvability of the linear-algebraic system (2.92). The main object of this section is to give a sufficient condition in order to get the invertibility of the linear algebraic system (2.92). To achieve this, first we state the following lemma which estimates the eigenvalues of the elastic capacitance matrix of each scatterer in terms of its acoustic capacitance.

**Lemma 4.1.** Let \( \lambda_{\text{min}}^m \) and \( \lambda_{\text{max}}^m \) be the minimal and maximal eigenvalues of the elastic capacitance matrices \( C_m \), for \( m = 1, 2, \ldots, M \). Denote by \( C_a \) the capacitance of each scatterer in the acoustic case,\(^8\) then we have the following estimate:

\[
\mu C_a \leq \lambda_{\text{min}}^{m} \leq \lambda_{\text{max}}^{m} \leq (\lambda + 2\mu) C_a, \quad \text{for} \quad m = 1, 2, \ldots, M. \tag{4.1}
\]

**Proof of Lemma 4.1.** Proof of this Lemma follows as in [21, Lemma 6.3.6]. See also [22, Lemma 10]. \( \Box \)

Now, we prove the main lemma of this section.

**Lemma 4.2.** The matrix \( B \) is invertible and the solution vector \( \hat{Q} \) of (2.92) satisfies the estimate:

\[
\sum_{m=1}^{M} \| \hat{Q}_m \|_2^2 \leq 4 \left( \frac{M}{\min_{m=1} \lambda_{\text{min}}^{m}} - \frac{3M}{\max_{m=1} \lambda_{\text{max}}^{m}} \right)^2 \sum_{m=1}^{M} \| U^i(z_m) \|_2^2, \tag{4.2}
\]

if we consider \( \max_{1 \leq m \leq M} \lambda_{\text{max}}^{m} \) with the positively assumed value

\[
t := \left[ \frac{1}{c_p^2} - 2\text{diam}(\Omega) \frac{1}{c_p} \frac{1-\left(\frac{1}{2} \kappa_{\nu} \text{diam}(\Omega)\right)^2}{1-\left(\frac{1}{2} \kappa_{\nu} \text{diam}(\Omega)\right)^{N_n}} \right] + \text{diam}(\Omega) \frac{1-\left(\frac{1}{2} \kappa_{\nu} \text{diam}(\Omega)\right)^{N_n}}{c_p} + \frac{1}{2^{n-1}}.
\]

**Proof of Lemma 4.2.** We can factorize \( B \) as \( B = -(I + B_a C)C^{-1} \) where \( C := \text{Diag}(C_1, C_2, \ldots, C_M) \in \mathbb{R}^{M \times M} \), \( I \) is the identity matrix and \( B_a := -C^{-1} - B \). Hence, the solvability of the system (2.92), depends on the existence of the inverse of \( (I + B_a C) \). We have \((I + B_a C) : \mathbb{C}^M \rightarrow \mathbb{C}^M \), so it is enough to prove the injectivity in order to prove its invertibility. For this purpose, let \( X, Y \) are vectors in \( \mathbb{C}^M \) and consider the system

\[
(I + B_a C)X = Y. \tag{4.3}
\]

Let \((\cdot)^{\text{real}}\) and \((\cdot)^{\text{img}}\) denotes the real and the imaginary parts of the corresponding complex number/vector/matrix. Now, the following can be written from (4.3):

\[
(I + B_n^{\text{real}} C)X^{\text{real}} - B_n^{\text{imag}} C X^{\text{imag}} = Y^{\text{real}}, \tag{4.4}
\]

\[
(I + B_n^{\text{real}} C)X^{\text{imag}} + B_n^{\text{imag}} C X^{\text{real}} = Y^{\text{imag}}, \tag{4.5}
\]

which leads to

\[
\langle (I + B_n^{\text{real}} C)X^{\text{real}}, CX^{\text{real}} \rangle - \langle B_n^{\text{imag}} C X^{\text{imag}}, CX^{\text{real}} \rangle = \langle Y^{\text{real}}, CX^{\text{real}} \rangle, \tag{4.6}
\]

\[
\langle (I + B_n^{\text{real}} C)X^{\text{imag}}, CX^{\text{imag}} \rangle + \langle B_n^{\text{imag}} C X^{\text{real}}, CX^{\text{imag}} \rangle = \langle Y^{\text{imag}}, CX^{\text{imag}} \rangle. \tag{4.7}
\]

By summing up (4.6) and (4.7) will give

\[
\langle X^{\text{real}}, CX^{\text{real}} \rangle + \langle B_n^{\text{real}} C X^{\text{real}}, CX^{\text{real}} \rangle + \langle X^{\text{imag}}, CX^{\text{imag}} \rangle + \langle B_n^{\text{real}} C X^{\text{imag}}, CX^{\text{imag}} \rangle = \langle Y^{\text{real}}, CX^{\text{real}} \rangle + \langle Y^{\text{imag}}, CX^{\text{imag}} \rangle. \tag{4.8}
\]

Indeed,

\[
\langle B_n^{\text{imag}} C X^{\text{imag}}, CX^{\text{real}} \rangle = \langle CX^{\text{imag}}, B_n^{\text{imag}} C X^{\text{real}} \rangle = \langle CX^{\text{imag}}, B_n^{\text{imag}} C X^{\text{real}} \rangle = \langle B_n^{\text{imag}} C X^{\text{real}}, CX^{\text{imag}} \rangle.
\]

\(^8\)Recall that, for \( m = 1, \ldots, M \), \( C_m := \int_{\partial D_m} \sigma_m(s)ds \) and \( \sigma_m \) is the solution of the integral equation of the first kind \( \int_{\partial D_m} \sigma_m(s)ds = 1, \ t \in \partial D_m \), see [11].
We can observe that, the right-hand side in (4.8) does not exceed
\[
(X^{\text{real}}, CX^{\text{real}})^{1/2} \langle Y^{\text{real}}, CY^{\text{real}} \rangle^{1/2} + (X^{\text{imag}}, CX^{\text{imag}})^{1/2} \langle Y^{\text{imag}}, CY^{\text{imag}} \rangle^{1/2} \\
\leq 2 (X^{\downarrow}_{\|\cdot\|}, (CX)^{\downarrow}_{\|\cdot\|})^{1/2} \langle Y^{\downarrow}_{\|\cdot\|}, (CY)^{\downarrow}_{\|\cdot\|} \rangle^{1/2}.
\]
(4.9)
Here \(W_{m}^{\downarrow}_{\|\cdot\|} := \|W_{m}^{\text{real}}\|^{2} + \|W_{m}^{\text{imag}}\|^{2} \rangle = \|W_{m}\|_{2}\), for \(W = X, Y\) and \(m = 1, \ldots, M\). Consider the second term in the left-hand side of (4.8). Using the mean value theorem for harmonic functions we deduce
\[
\langle B_{n}^{\text{real}} CX^{\text{real}}, CX^{\text{real}} \rangle = \sum_{1 \leq j, k \leq m} X_{n}^{\text{real}} C_{m}^{T} \frac{1}{\|B_{j}\| \|B_{k}\|} \int_{B_{j}} \int_{B_{k}} \Phi_{0}(x, y) \, dx \, dy \rangle \;
C_{j} X_{j}^{\text{real}},
\]
Similarly, if we consider the fourth term in the left-hand side of (4.8), we deduce
\[
\langle B_{n}^{\text{imag}} CX^{\text{imag}}, CX^{\text{imag}} \rangle = \sum_{1 \leq j, k \leq m} X_{n}^{\text{imag}} C_{m}^{T} \frac{1}{\|B_{j}\| \|B_{k}\|} \int_{B_{j}} \int_{B_{k}} \Phi_{0}(x, y) \, dx \, dy \rangle \;
C_{j} X_{j}^{\text{imag}},
\]
where \(t := \left[ \frac{1}{p} - 2 \text{diam}(\Omega) \frac{1}{\omega^{2}} \left( \frac{1}{4} \kappa_{\omega} \text{diam}(\Omega) + \frac{1}{2 \pi^{n}} \right) - \text{diam}(\Omega) \frac{1}{\omega^{2}} \left( \frac{1}{4} \kappa_{\omega} \text{diam}(\Omega) + \frac{1}{2 \pi^{n}} \right) \right] \)
assumed to be positive, \(\Phi_{0}(x, y) := 1/(4\pi|x-y|), x \neq y\) and \(B_{j}^{(m)} := \{ x : |x-z_{m}| < d/2 \}, m = 1, \ldots, M\), are non-overlapping balls of radius \(d/2\) with centers at \(z_{m}\), and \(|B_{j}^{(m)}| = \pi d^{3}/6\) are the volumes of the balls. Also, we use the notation \(B_{d}\) to denote the balls of radius \(d/2\) with the center at the origin.
Indeed, we can write \(\Gamma^{\omega}(z_{m}, z_{j})\) from (2.3) as,
\[
\Gamma^{\omega}(z_{m}, z_{j}) = \left( \frac{1}{4\pi |z_{m} - z_{j}|} \frac{1}{2} \left[ \frac{1}{c_{a}} + \frac{1}{c_{p}} \right] I + \frac{1}{2} \left[ \frac{1}{c_{a}} - \frac{1}{c_{p}} \right] \right) \frac{1}{|z_{m} - z_{j}|} \odot (z_{m} - z_{j})
\]
\[
+ \sum_{l=1}^{\infty} \frac{l!}{l(l+2)} \frac{1}{\omega^{2}} \left( (l+1)\kappa_{\omega}^{l+2} + \kappa_{p}^{l+2} \right) |z_{m} - z_{j}|^{l} I,
\]
\[
- \sum_{l=1}^{\infty} \frac{l!}{l(l+2)} \frac{1}{\omega^{2}} \left( \kappa_{\omega}^{l+2} - \kappa_{p}^{l+2} \right) |z_{m} - z_{j}|^{l-2} (z_{m} - z_{j}) \odot (z_{m} - z_{j}) \right) \),
\]
(4.10)
from which, we get the required result by estimating \(\Gamma^{\omega}(z_{m}, z_{j})\). Notice that
\[
|b_{l}| \leq \frac{1}{2} \left[ \frac{1}{c_{a}^{2}} - \frac{1}{c_{p}^{2}} \right] \text{ and}
\]
\[
|c_{1}r + c_{2}r| \leq \sum_{l=1}^{\infty} \frac{1}{(l-1)!l(l+2)} \frac{1}{\omega^{2}} \left( 2\kappa_{\omega}^{l+2} + \kappa_{p}^{l+2} \right) |z_{m} - z_{j}|^{l}.
\]
[By recalling \( N_\Omega = [2 \text{diam}(\Omega) \max\{\kappa_{w^+},\kappa_{p^+}\} e^2] \) and using Lemma 2.6]

\[
\begin{align*}
&\leq \text{diam}(\Omega) \left[ \frac{2\omega}{c_p^2} \sum_{l=1}^{N_\Omega} \left( \frac{1}{2} \kappa_w \text{diam}(\Omega) \right)^{l-1} + \sum_{l=N_\Omega+1}^{\infty} \frac{1}{2^{l-1}} \right] \\
&\quad + \frac{\omega}{c_p^3} \sum_{l=1}^{N_\Omega} \left( \frac{1}{2} \kappa_p \text{diam}(\Omega) \right)^{l-1} + \sum_{l=N_\Omega+1}^{\infty} \frac{1}{2^{l-1}} \right] \\
&= \text{diam}(\Omega) \left[ \frac{2\omega}{c_p^2} \left( 1 - \left( \frac{1}{2} \kappa_w \text{diam}(\Omega) \right)^{N_\Omega} \right) + \frac{1}{2^{N_\Omega-1}} \right] \\
&\quad + \frac{\omega}{c_p^3} \left( 1 - \left( \frac{1}{2} \kappa_p \text{diam}(\Omega) \right) + \frac{1}{2^{N_\Omega-1}} \right),
\end{align*}
\]

Let \( \Omega \) be a large ball with radius \( R \). Also let \( \Omega_1 \subset \Omega \) be a ball with fixed radius \( r (\leq R) \), which consists of all our small obstacles \( D_m \) and also the balls \( B^{(m)} \), for \( m = 1, \ldots, M \).

Let \( \Upsilon_{\text{real}}(x) \) and \( \Upsilon_{\text{img}}(x) \) be piecewise constant functions defined on \( \mathbb{R}^3 \) as

\[
\Upsilon_{\text{real}}(x) = \begin{cases} 
\mathcal{C}_m \chi_{m}^{\text{real}}(y) & \text{in } B^{(m)}, \quad m = 1, \ldots, M, \\
0 & \text{otherwise}.
\end{cases}
\]

Then

\[
\langle \mathbf{B}_n^{\text{real}} \mathbf{C} \chi_{\text{real}}, \mathbf{C} \chi_{\text{real}} \rangle \geq \frac{36t}{\pi^2} \left( \int_{\Omega} \int_{\mathbb{R}^3} \Phi_0(x,y) \Upsilon_{\text{real}}^T(x) \Upsilon_{\text{real}}(y) \, dx \, dy \\
- \sum_{m=1}^{M} \left| \mathcal{C}_m \chi_{m}^{\text{real}} \right|^2 \int_{B^{(m)}} \int_{B^{(m)}} \Phi_0(x,y) \, dx \, dy \right) \tag{4.12}
\]

\[
\langle \mathbf{B}_n^{\text{real}} \mathbf{C} \chi_{\text{img}}, \mathbf{C} \chi_{\text{img}} \rangle \geq \frac{36t}{\pi^2} \left( \int_{\Omega} \int_{\mathbb{R}^3} \Phi_0(x,y) \Upsilon_{\text{img}}^T(x) \Upsilon_{\text{img}}(y) \, dx \, dy \\
- \sum_{m=1}^{M} \left| \mathcal{C}_m \chi_{m}^{\text{img}} \right|^2 \int_{B^{(m)}} \int_{B^{(m)}} \Phi_0(x,y) \, dx \, dy \right) \tag{4.13}
\]

Applying the mean value theorem to the harmonic function \( \frac{1}{4\pi|x-y|} \), as done in [20, p:109-110], we have the following estimate

\[
\int_{B^{(m)}} \int_{B^{(m)}} \Phi_0(x,y) \, dx \, dy \leq \frac{1}{4\pi} \int_{B_R} \int_{B_R} \frac{1}{|x-y|} \, dx \, dy \leq \frac{\pi d^3}{60}. \tag{4.14}
\]

Consider the first term in the right-hand side of (4.12), denote it by \( A_R^{\text{real}} \), then by Green’s theorem

\[
A_R^{\text{real}} := \int_{\Omega} \int_{\Omega} \Phi_0(x,y) \Upsilon_{\text{real}}^T(x) \Upsilon_{\text{real}}(y) \, dx \, dy \tag{4.15}
\]

\[
= \left( \int_{\Omega} \left| \nabla_x \int_{\Omega} \Phi_0(x,y) \Upsilon_{\text{real}}(y) \, dy \right|^2 \right)_{=:\text{B}_n^{\text{real}} \geq 0} \left( \int_{\partial \mathcal{C}_R^{\text{real}}} \left( \frac{\partial}{\partial n_x} \int_{\Omega} \Phi_0(x,y) \Upsilon_{\text{real}}(y) \, dy \right) \, dS_x.\right)
\]
We have
\[
C_{R} = \int_{\Omega} \left( \int_{\Omega} \frac{\partial}{\partial \nu_x} \Phi_0(x,y) \mathcal{T}_{\text{real}}(y) \, dy \right) \left( \int_{\Omega} \Phi_0(x,y) \mathcal{T}_{\text{real}}(y) \, dy \right) \, dS_x
= \int_{\Omega} \left( \int_{\Omega} \frac{\partial}{\partial \nu_x} \Phi_0(x,y) \mathcal{T}_{\text{real}}(y) \, dy \right) \left( \int_{\Omega} \Phi_0(x,y) \mathcal{T}_{\text{real}}(y) \, dy \right) \, dS_x
= \int_{\Omega} \left( \frac{-(x-y)}{4\pi|x-y|^3} \mathcal{T}_{\text{real}}(y) \right) \left( \int_{\Omega} \frac{1}{4\pi|x-y|} \mathcal{T}_{\text{real}}(y) \, dy \right) \, dS_x,
\]
which gives the following estimate;
\[
|C_{R}| \leq \frac{1}{16\pi^2} \int_{\partial \Omega} \frac{1}{|R-r|^3} \left( \int_{\Omega} |\mathcal{T}_{\text{real}}(y)| \, dy \right)^2 \, dS_x
\leq \frac{1}{16\pi^2} \frac{1}{(R-r)^3} \int_{\partial \Omega} |\Omega| \left| \mathcal{T}_{\text{real}} \right|_{L^2(\Omega)}^2 \, dS_x
= \frac{r^3}{12\pi(R-r)^3} \sum_{m=1}^{M} |C_m X_{m}^{\text{real}}|^2 |\Omega|
= \frac{R^2 r^3}{3(R-r)^3} \sum_{m=1}^{M} |C_m X_{m}^{\text{real}}|^2.
\]
(4.17)

Substitution of (4.17) in (4.15) gives
\[
\int_{\Omega} \int_{\Omega} \Phi_0(x,y) \mathcal{T}_{\text{real}}(x) \mathcal{T}_{\text{real}}(y) \, dx \, dy \geq \int_{\Omega} \left| \nabla_x \int_{\Omega} \Phi_0(x,y) \mathcal{T}_{\text{real}}(y) \, dy \right|^2 \, dx - \frac{R^2 r^3}{3(R-r)^3} \sum_{m=1}^{M} |C_m X_{m}^{\text{real}}|^2.
\]
(4.18)

By considering the first term in the right-hand side of (4.13), and following the same procedure as mentioned in (4.15), (4.16) and (4.17), we obtain
\[
\int_{\Omega} \int_{\Omega} \Phi_0(x,y) \mathcal{T}_{\text{img}}(x) \mathcal{T}_{\text{img}}(y) \, dx \, dy \geq \int_{\Omega} \left| \nabla_x \int_{\Omega} \Phi_0(x,y) \mathcal{T}_{\text{img}}(y) \, dy \right|^2 \, dx - \frac{R^2 r^3}{3(R-r)^3} \sum_{m=1}^{M} |C_m X_{m}^{\text{img}}|^2.
\]
(4.19)

Under our assumption \( t > 0 \), (4.12), (4.13) (4.14), (4.18) and (4.19) lead to
\[
\langle B^\text{real}_n, C X^{\text{real}} \rangle \geq \frac{36t}{\pi^2 d^5} \left( \int_{\Omega} \left| \nabla_x \int_{\Omega} \Phi_0(x,y) \mathcal{T}_{\text{real}}(y) \, dy \right|^2 \, dx - \left[ \frac{R^2 r^3}{3(R-r)^3} + \frac{\pi d^5}{60} \right] \sum_{m=1}^{M} |C_m X_{m}^{\text{real}}|^2 \right),
\]
(4.20)
\[
\langle B^\text{img}_n, C X^{\text{img}} \rangle \geq \frac{36t}{\pi^2 d^5} \left( \int_{\Omega} \left| \nabla_x \int_{\Omega} \Phi_0(x,y) \mathcal{T}_{\text{img}}(y) \, dy \right|^2 \, dx - \left[ \frac{R^2 r^3}{3(R-r)^3} + \frac{\pi d^5}{60} \right] \sum_{m=1}^{M} |C_m X_{m}^{\text{img}}|^2 \right).
\]
(4.21)

Then (4.8), (4.20) and (4.21) imply
\[
\left( \frac{M}{\min_{m=1}^{M} \lambda_{m}^{\text{min}} \lambda_{m}^{\text{max}}} \right) \leq 2 \left( \frac{M}{\max_{m=1}^{M} \lambda_{m}^{\text{max}} \lambda_{m}^{\text{max}}} \right) \left( \sum_{m=1}^{M} \left| X_m \right|^2 \right)^{1/2} \left( \sum_{m=1}^{M} \left| Y_m \right|^2 \right)^{1/2}.
\]
(4.22)
As we have $R$ arbitrary, by tending $R$ to $\infty$, we can write (4.22) as
\[
\left( \min_{m=1}^{M} \lambda_{\text{eig}_{m}}^{\min} - \frac{3t}{5 \pi d} \max_{m=1}^{M} \lambda_{\text{eig}_{m}}^{\max} \right) ^{2} \left( \sum_{m=1}^{M} \|X_{m}\|_{2} \right) \leq 2 \left( \max_{m=1}^{M} \lambda_{\text{eig}_{m}}^{\max} \right) \left( \sum_{m=1}^{M} \|Y_{m}\|_{2} \right) ^{\frac{1}{2}} \left( \sum_{m=1}^{M} \|Y_{m}\|_{2} \right) ^{\frac{1}{2}}.
\]
(4.23)

which yields
\[
\sum_{m=1}^{M} \|X_{m}\|_{2} \leq 4 \left( \min_{m=1}^{M} \lambda_{\text{eig}_{m}}^{\min} - \frac{3t}{5 \pi d} \max_{m=1}^{M} \lambda_{\text{eig}_{m}}^{\max} \right) ^{-2} \left( \max_{m=1}^{M} \lambda_{\text{eig}_{m}}^{\max} \right) ^{2} \sum_{m=1}^{M} \|Y_{m}\|_{2}^{2}.
\]
(4.24)

Thus, if \( \left( \max_{1 \leq m \leq M} \lambda_{\text{eig}_{m}}^{\max} \right) < t^{-1} \left( \frac{5 \pi d}{3 \mu} \min_{1 \leq m \leq M} \lambda_{\text{eig}_{m}}^{\min} \right) \) then the matrix $B$ in algebraic system (2.92) is invertible and the estimate (4.23) and so (4.2) holds.

**Corollary 4.3.** If $(\lambda + 2 \mu)^{2} \left( \max_{1 \leq m \leq M} C_{m}^{\alpha} \right) ^{2} < t^{-1} \left( \frac{5 \pi d}{3 \mu} \min_{1 \leq m \leq M} C_{m}^{\alpha} \right) ^{2}$, then the matrix $B$ is invertible and the solution vector $\tilde{Q}$ of (2.92) satisfies the estimate:
\[
\sum_{m=1}^{M} \|\tilde{Q}_{m}\|_{2} \leq 2 \left( 1 - \frac{3t}{5 \pi} \mu \frac{M}{d} \min_{m=1}^{M} C_{m}^{\alpha} \right) \left( \sum_{m=1}^{M} \|Y_{m}\|_{2} \right) \left( \max_{m=1}^{M} C_{m}^{\alpha} \right) \frac{M}{\min_{m=1}^{M} C_{m}^{\alpha}} \left( \sum_{m=1}^{M} \|Y_{m}\|_{2} \right) ^{\frac{1}{2}} \left( \sum_{m=1}^{M} \|Y_{m}\|_{2} \right) ^{\frac{1}{2}}.
\]
(4.25)

**Proof of Corollary 4.3.** Let us assume the condition $(\lambda + 2 \mu)^{2} \left( \max_{1 \leq m \leq M} C_{m}^{\alpha} \right) ^{2} < t^{-1} \left( \frac{5 \pi d}{3 \mu} \min_{1 \leq m \leq M} C_{m}^{\alpha} \right) ^{2}$, then from Lemma 4.1 the sufficient condition of Lemma 4.2 is satisfied and hence (4.2) holds. Now, by applying the norm inequalities to (4.2), we obtain
\[
\sum_{m=1}^{M} \|\tilde{Q}_{m}\|_{2} \leq 2 \left( \min_{m=1}^{M} \lambda_{\text{eig}_{m}}^{\min} - \frac{3t}{5 \pi d} \max_{m=1}^{M} \lambda_{\text{eig}_{m}}^{\max} \right) ^{-2} \left( \max_{m=1}^{M} \lambda_{\text{eig}_{m}}^{\max} \right) ^{2} \sum_{m=1}^{M} \|Y_{m}\|_{2}^{2}.
\]
(4.26)

Now, again by applying Lemma 4.1 to the above inequality (4.26) gives the result (4.25).

---

## 5. Appendix

Here we provide results concerning the single layer potential used in the previous sections.

### 5.1. The case of a single obstacle

Let us consider a single obstacle $D_{\epsilon} := \epsilon B + z$ with Lipschitz boundary. Then define the operator $S_{D_{\epsilon}} : L^{2}(\partial D_{\epsilon}) \to H^{1}(\partial D_{\epsilon})$ by
\[
(S_{D_{\epsilon}} \psi)(x) := \int_{\partial D_{\epsilon}} \Gamma(x, y) \psi(y) dy.
\]
(5.1)

**Lemma 5.1.** The operator $S_{D_{\epsilon}}$ is invertible.

**Proof of Lemma 5.1.** Proof of this Lemma follows as the one of Proposition 2.1.

---

**Lemma 5.2.** Let $\phi \in H^{1}(\partial D_{\epsilon})$ and $\psi \in L^{2}(\partial D_{\epsilon})$. Then,
\[
S_{D_{\epsilon}} \psi = \epsilon (S_{B} \tilde{\psi})^{\gamma},
\]
(5.2)
\[ S_{D_s}^{-1}\phi = \epsilon^{-1}(S_B^{-1}\phi)^\vee \] (5.3)

and
\[
\left\| S_{D_s}^{-1} \right\|_{L(H^1(\partial D_s), L^2(\partial D_s))} \leq \epsilon^{-1} \left\| S_B^{-1} \right\|_{L(H^1(\partial B), L^2(\partial B))}
\] (5.4)

with \( S_B^{\vee}(\xi) := \int_{\partial B} \Gamma^\omega(\xi, \eta)\psi(\eta)d\eta \).

**Proof of Lemma 5.2.**

- We have,
\[
S_{D_s}\psi(x) = \int_{\partial D_s} \Gamma^\omega(x, y)\psi(y)dy
= \int_{\partial B} \frac{1}{\epsilon} \Gamma^\omega(\xi, \eta)\psi(\epsilon\eta + z)\epsilon^2 d\eta
= \epsilon S_B^{\vee}\psi(\xi).
\]

The above gives us (5.2).

- The following equalities, using (5.2),
\[
S_{D_s}(S_B^{-1}\phi)^\vee = \epsilon (S_B^{\vee}S_B^{-1}\phi)^\vee = \epsilon \phi^\vee = \epsilon \phi
\]
provides us (5.3).

- We have from the estimate,
\[
\left\| S_{D_s}^{-1} \right\|_{L(H^1(\partial D_s), L^2(\partial D_s))} := \sup_{\phi(\neq 0) \in H^1(\partial D_s)} \frac{\left\| S_{D_s}^{-1}\phi \right\|_{L^2(\partial D_s)}}{\left\| \phi \right\|_{H^1(\partial D_s)}}
\leq (2.18)(2.19) \sup_{\phi(\neq 0) \in H^1(\partial D_s)} \frac{\phi^{-1} \left\| (S_{D_s}^{-1}\phi)^\vee \right\|_{L^2(\partial B)}}{\left\| \phi \right\|_{H^1(\partial B)}}
= (5.3) \sup_{\phi(\neq 0) \in H^1(\partial D_s)} \frac{\epsilon^{-1} \left\| S_B^{-1}\phi \right\|_{L^2(\partial B)}}{\left\| \phi \right\|_{H^1(\partial B)}}
= \epsilon^{-1} \left\| S_B^{-1} \right\|_{L(H^1(\partial B), L^2(\partial B))}.
\]

**Lemma 5.3.** The operator norm of the inverse of \( S_{D_*} : L^2(\partial D_*) \to H^1(\partial D_*) \) has the following upper bound:
\[
\left\| S_{D_*}^{-1} \right\|_{L(H^1(\partial D_*), L^2(\partial D_*))} \leq C_0 \epsilon^{-1},
\] (5.5)

with \( C_0 := \frac{2\pi \left\| S_B^{-1} \right\|_{L(H^1(\partial B), L^2(\partial B))}}{2\pi \max \left\{ \frac{1}{\epsilon} \left\| S_B^{-1} \right\|_{L(H^1(\partial B), L^2(\partial B))}, \frac{1}{(1+2\omega)\epsilon} \left\| S_B^{-1} \right\|_{L(H^1(\partial B), L^2(\partial B))} \right\} \).

Here we should mention that if \( \epsilon \leq \frac{\pi}{\max \left\{ \frac{1}{\epsilon} \left\| S_B^{-1} \right\|_{L(H^1(\partial B), L^2(\partial B))}, \frac{1}{(1+2\omega)\epsilon} \left\| S_B^{-1} \right\|_{L(H^1(\partial B), L^2(\partial B))} \right\}} \), then \( C_0 \) is bounded by \( 2 \left\| S_B^{-1} \right\|_{L(H^1(\partial B), L^2(\partial B))} \), which is a universal constant depending only on \( \partial B \) through its Lipschitz character.

**Proof of Lemma 5.3.** To estimate the operator norm of \( S_{D_*}^{-1} \) we decompose \( S_{D_*} := S_{D_*}^{\omega} = S_{D_*}^{\omega*} + S_{D_*}^{\omega*} \) in to two parts \( S_{D_*}^{\omega*} \) (independent of \( \omega \)) and \( S_{D_*}^{\omega*} \) (dependent of \( \omega \)) given by
\[
S_{D_*}^{\omega*}\psi(x) := \int_{\partial D_*} \Gamma^0(x, y)\psi(y)dy.
\] (5.6)
\[ S_{D_i}^{\text{d}} \psi(x) := \int_{D_i} [\Gamma^\omega(x,y) - \Gamma^0(x,y)] \psi(y) dy. \] (5.7)

With this definition, \( S_{D_i}^{\text{d}} : L^2(\partial D_i) \to H^1(\partial D_i) \) is invertible, see [19, 24–26]. Hence, \( S_{D_i} = S_{D_i}^{\text{d}} \left( I + S_{D_i}^{\text{d}}^{-1} S_{D_i}^{\text{d}} \right) \) and so

\[
\left\| S_{D_i}^{-1} \right\|_{L(H^1(\partial D_i), L^2(\partial D_i))} = \left\| \left( I + S_{D_i}^{\text{d}}^{-1} S_{D_i}^{\text{d}} \right)^{-1} S_{D_i}^{-1} \right\|_{L(H^1(\partial D_i), L^2(\partial D_i))} \\
\leq \left\| \left( I + S_{D_i}^{\text{d}}^{-1} S_{D_i}^{\text{d}} \right)^{-1} \right\|_{L(L^2(\partial D_i), L^2(\partial D_i))} \left\| S_{D_i}^{\text{d}}^{-1} \right\|_{L(H^1(\partial D_i), L^2(\partial D_i))}. \] (5.8)

So, to estimate the operator norm of \( S_{D_i}^{-1} \) one needs to estimate the operator norm of \( \left( I + S_{D_i}^{\text{d}}^{-1} S_{D_i}^{\text{d}} \right)^{-1} \), in particular one needs to have the knowledge about the operator norms of \( S_{D_i}^{\text{d}}^{-1} \) and \( S_{D_i}^{\text{d}} \) to apply the Neumann series. For that purpose, we can estimate the operator norm of \( S_{D_i}^{\text{d}}^{-1} \) from (5.4) by

\[
\left\| S_{D_i}^{\text{d}}^{-1} \right\|_{L(H^1(\partial D_i), L^2(\partial D_i))} \leq \epsilon^{-1} \left\| S_{B}^{\text{d}}^{-1} \right\|_{L(H^1(\partial B), L^2(\partial B))}. \] (5.9)

Here \( S_{B}^{\text{d}} \hat{\psi}(\xi) := \int_{\partial B} \Gamma^0(\xi,\eta) \hat{\psi}(\eta) d\eta \). From the definition of the operator \( S_{D_i}^{\text{d}} \) in (5.7), we deduce that

\[
S_{D_i}^{\text{d}} \psi(x) = \epsilon \int_{\partial B} [\Gamma^\omega(\xi,\eta) - \Gamma^0(\xi,\eta)] \hat{\psi}(\eta) d\eta. \] (5.10)

Hence,

\[
\left\| S_{D_i}^{\text{d}} \psi(x) \right\| = \left\| \epsilon \int_{\partial B} \frac{1}{4\pi} \sum_{l=1}^{\infty} \frac{\epsilon l}{\pi(l+2)} \omega \beta \left( l(l+1) \gamma \right) \Psi_{l}^{\text{d}} \left( \frac{1}{\sqrt{2}} \right) \delta \right\|_{L^2(D)} \leq \left\| \epsilon \int_{\partial B} \frac{1}{4\pi} \sum_{l=1}^{\infty} \frac{\epsilon l}{\pi(l+2)} \omega \beta \left( l(l+1) \gamma \right) \Psi_{l}^{\text{d}} \left( \frac{1}{\sqrt{2}} \right) \delta \right\|_{L^2(D)}. \]
\[ \leq \frac{\epsilon^2 \omega}{2\pi} \| \hat{\psi} \|_{L^2(\partial B)} | \partial B | \frac{1}{2} \left[ \frac{2}{c_s^2} + \frac{1}{c_p^2} \right], \text{for } \epsilon \leq \frac{\min \{ c_s, c_p \}}{\omega_{\text{max}} \max_m \text{diam}(B_m)}. \]  

(5.11)

We set \( C_1 := \frac{|\partial B|}{2\pi} \left[ \frac{2}{c_s^2} + \frac{1}{c_p^2} \right] \). From this we get,

\[
||S_{D_0}^c \hat{\psi}||_{L^2(\partial D_0)}^2 = \int_{\partial D_0} |S_{D_0}^c \hat{\psi}(x)|^2 \, dx 
\]

\[
\leq \int_{\partial D_0} \left[ C_1 \omega |\hat{\psi}|_{L^2(\partial B)}^2 \right] \, dx = C_1 \omega^2 |\partial B| \| \hat{\psi} \|^2_{L^2(\partial B)},
\]

which gives us

\[
||S_{D_0}^c \hat{\psi}||_{L^2(\partial D_0)} \leq C_1 \omega^3 |\partial B|^\frac{1}{2} \| \hat{\psi} \|_{L^2(\partial B)}. \]

(5.12)

By making use of (5.10), the tangential derivative w.r.t basic vector \( T \) (\( T_1 \) or \( T_2 \)) is given by

\[
\frac{\partial}{\partial T} S_{D_0}^c \hat{\psi}(x) = \int_{\partial B} \nabla_\xi [\Gamma^{\omega \xi}(\xi, \eta) - \Gamma^{0}(\xi, \eta)] \cdot T_\xi \hat{\psi}(\eta) \, d\eta.
\]

\[
= \frac{1}{4\pi} \sum_{l=2}^{\infty} \frac{\epsilon^2}{l!(l+2)} \frac{l-1}{\omega^2} \left[ \int_{\partial B} ((l+1)k_s^{l+2} + k_p^{l+2}) |\xi - \eta|^{-3} (\xi - \eta) \otimes \mathbf{I} \cdot T_\xi \hat{\psi}(\eta) \, d\eta - \int_{\partial B} (k_s^{l+2} - k_p^{l+2}) (l-3)(\xi - \eta)|^{-3} (\xi - \eta) + \mathbf{I} \otimes (\xi - \eta) + (\xi - \eta) \otimes \mathbf{I} \cdot T_\xi \hat{\psi}(\eta) \, d\eta \right]
\]

\[
= \frac{1}{4\pi} \sum_{l=2}^{\infty} \frac{\epsilon^2}{l!(l+2)} \frac{l-1}{\omega^2} \left[ \int_{\partial B} ((l+1)k_s^{l+2} + k_p^{l+2}) |\xi - \eta|^{-3} (\xi - \eta) \otimes \mathbf{I} \cdot T_\xi \hat{\psi}(\eta) \, d\eta - \int_{\partial B} (k_s^{l+2} - k_p^{l+2}) (l-3)(\xi - \eta)|^{-3} (\xi - \eta) + \mathbf{I} \otimes (\xi - \eta) + (\xi - \eta) \otimes \mathbf{I} \cdot T_\xi \hat{\psi}(\eta) \, d\eta \right].
\]

(5.13)

It provides us

\[
\left| \frac{\partial}{\partial T} S_{D_0}^c \hat{\psi}(x) \right| \leq \frac{\epsilon^2}{4\pi} \left[ \sum_{l=3}^{\infty} \frac{\epsilon^{l-2}}{(l+2)!} \frac{l-1}{\omega^2} \left( 2k_s^{l+2} + k_p^{l+2} \right) \int_{\partial B} |\xi - \eta|^{-2} |\hat{\psi}(\eta)| \, d\eta + \frac{\left( 6k_s^{l+2} + 4k_p^{l+2} \right)}{8\omega^2} \int_{\partial B} |\hat{\psi}(\eta)| \, d\eta \right]
\]

\[
\leq \frac{\epsilon^2}{4\pi} \left| \hat{\psi} \right|_{L^2(\partial B)} \left[ \sum_{l=3}^{\infty} \frac{\epsilon^{l-2}}{(l+2)!} \frac{l-1}{\omega^2} \left( 2k_s^{l+2} + k_p^{l+2} \right) \| \xi - \eta \|_{L^2(\partial B)}^{-2} |\partial B|^{\frac{2}{l-2}} + \frac{\left( 6k_s^{l+2} + 4k_p^{l+2} \right)}{8\omega^2} |\partial B|^{\frac{1}{2}} \right]
\]

(Since, \( ||x||_{L^2(D)} \leq ||x||_{L^2(\partial B)} |D|^{\frac{1}{2}} \))

\[
= \frac{\epsilon^2}{4\pi} \left| \hat{\psi} \right|_{L^2(\partial B)} |\partial B|^{\frac{1}{2}} \left[ \frac{\omega^2}{4c_s^4} \left( 1 + 8 \sum_{l=2}^{\infty} \frac{\epsilon^{l-2}}{(l+2)!} \frac{l-1}{\omega^2} k_s^{l+2} \| \xi - \eta \|_{L^2(\partial B)}^{-2} |\partial B|^{\frac{2}{l-2}} \right)
\]

\[
+ \frac{\omega^2}{4c_p^4} \left( 1 + 4 \sum_{l=2}^{\infty} \frac{\epsilon^{l-2}}{(l+2)!} k_p^{l+2} \| \xi - \eta \|_{L^2(\partial B)}^{-2} |\partial B|^{\frac{2}{l-2}} \right) \right]
\]

(Since, \( k_s = \omega/c_s, k_p = \omega/c_p \))

\[
\leq \frac{\epsilon^2}{4\pi} \left| \hat{\psi} \right|_{L^2(\partial B)} |\partial B|^{\frac{1}{2}} \left[ \frac{\omega^2}{4c_s^4} \left( 1 + 8 \sum_{l=2}^{\infty} \left( \frac{1}{2} \epsilon k_s \| \xi - \eta \|_{L^2(\partial B)} |\partial B|^{\frac{1}{2}} \right)^l \right)
\]

\[
+ \frac{\omega^2}{4c_p^4} \left( 1 + 4 \sum_{l=0}^{\infty} \left( \frac{1}{2} \epsilon k_p \| \xi - \eta \|_{L^2(\partial B)} |\partial B|^{\frac{1}{2}} \right)^l \right) \right].
\]
\[
\phi \leq 2 \min \left\{ \frac{c_s, c_p}{\omega_{\max} \max_m \text{diam}(B_m)} \right\}
\]
The Foldy-Lax approximation of the elastic scattering by many small bodies

\[ \langle 5.9 \rangle \leq C_{5}(1 + 2\omega)\omega \epsilon \]

or

\[ \langle 5.17 \rangle \]

By substituting the above and (5.9) in (5.8), we obtain the required result (5.5).

REFERENCES


