Polynomial integration on regions defined by a triangle and a conic

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Motivation: optimization and optimal control

Many technical processes are described by partial differential equations.

$\Rightarrow$ Optimization problems in an infinite-dimensional setting

Given a system, we can exert some **control** over it, which has an influence on the **state** of the system. We want to do it in such a way to **minimize a certain functional**, while satisfying some **constraints**.

An **optimal control** minimizes the functional, and it satisfies the necessary optimality condition, which is a system of equations and inequalities. Analogy: for the problem $\min f(x)$, $x \in \mathbb{R}$, solve $f' = 0$, $f'' > 0$...

Example of optimal control: optimal cooking

Consider a typical Munich (Oktoberfest) mathematical problem.

We want to minimize the carbon foot-print of having a *wurst* ready to eat at a particular time of a particular day.

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- Control: fn. $heat(pos, time)$ of the grill (how much, when, where)
- Relation between heat induced and temperature: heat equation
- To minimize: tons of $\mathrm{CO}_2 +$ distance to optimal temperature
- Control constraints: otherwise, a solution could be "raise the stove temperature to $1000000000000K$ during $0.000000000001s$"
- State constraints: disallow final temperatures outside a suitable range

Discretization

The problem is approximated by a finite-dimensional problem, that we can solve. How?

1. Triangulate the region
2. Discretize the space of functions: consider polynomials of degree $\leq d$ on each triangle (finite-dimensional vector space)

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In the particular type of problem that we are interested in, the discretization process leads to the problem:

> To calculate accurately integrals of polynomial functions on regions $Triangle \cap \{f(x,y) \geq 0\}$ where $\deg f = d$.

$d = 1$ Error in $O(h^2)$ where h is the maximum diameter of the triangulation. The integrals are easy.

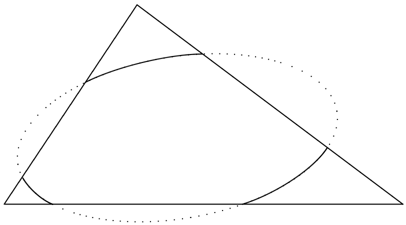
$d = 2$ Error in $O(h^3)$. Integrals not so easy: <u>today we will present a method</u>.

Statement and example

Calculate

$$\iint_{T\cap\{f\geq 0\}} (\phi_1 \cdot \phi_2)(x, y)\, dx\, dy\,, \quad \deg f = \deg \phi_i = 2$$

Example: integrate a degree 4 polynomial (monomial!) on the inner region:



The simplest case

If T is contained in one of the regions determined by f, the integral is 0 or the full triangle. In the second case, transform it affinely to $(0,0),(1,0),(1,1)$ and integrate

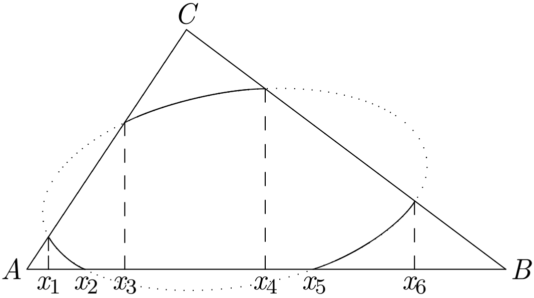
$$\int_0^1 \int_0^x g(x,y) \cdot |J| \, dy \, dx = |J| \sum_{i+j \leq 4} \frac{a_{ij}}{(j+1)(i+j+2)}$$

Note:

$$\iint_{T \cap \{f \geq 0\}} \quad + \quad \iint_{T \cap \{f \leq 0\}} \quad = \quad \iint_T$$

$\Rightarrow$ It suffices to integrate on a region or its complementary

The previous example



$$\int_{x_1}^{x_2} \int_{c_-(x)}^{l_{AC}(x)} x^i y^j \, dy \, dx \; + \text{ four more terms}$$

$c_-(x)$ involves a square root, which becomes part of the outer integrand.

Or: scale the ellipse into a circle, integrate sectors (polarly) and triangles.

The main problem

Even if we find a nice formula for this integral... we were given the relative position of the triangle and the conic.

There are "a few" possible relative positions. How to discern?

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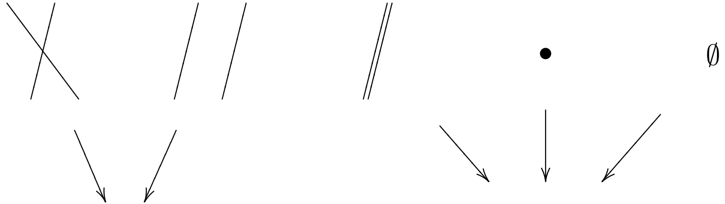
There are "a few" possible relative positions. How to discern?

Our approach: establish some base cases, cut the triangle into pieces that fall into those cases.

1. How to cut? Try to reduce the number of intersections, specially in the interior of the sides of the triangle. Try to avoid finding "complicated" points.
2. How to integrate? It would be nice (but rather irrelevant) if the base cases had simple formulas.

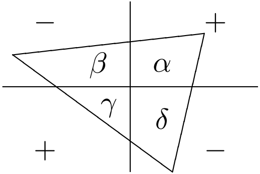
The degenerate case

If C is degenerate we can easily integrate.



integer combination of triangles $\qquad\qquad \iint_T$ or 0

Example:



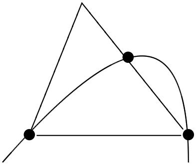
$$\alpha+\gamma=(\alpha+\beta+\gamma+\delta)-$$
$$-(\beta+\gamma)-(\delta+\gamma)+2\gamma$$

Basic tools

Fix a nondegenerate conic $C = \{f = 0\}$.

Definition: free segment

A segment is called free if it does not intersect C except possibly at the endpoints of the segment.



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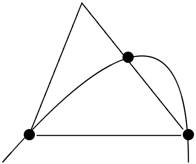


Two of the sides are free

1. A segment (or a line) cannot intersect C in more than two points.
2. A segment that joins two points of C is always free.

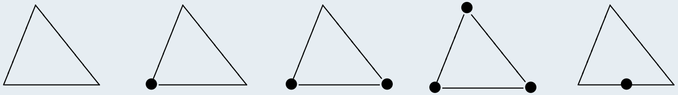
Calculating the intersections of a segment with C is easy. Thus, so it is determining if a segment is free.

Good triangles

Definition: good triangle

A triangle is called good if one of the following is true:

- its three sides are free; or
- $\partial T \cap C$ is one non-vertex point.



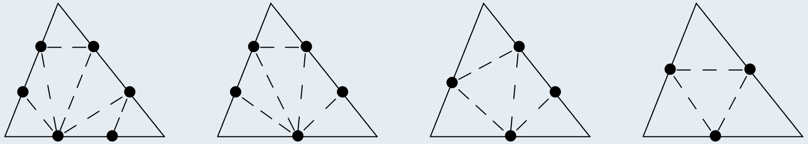
Integration formulas are quite simple if the triangle is good.

Theorem

Every triangle can be cut into 11 or less good triangles.

Case 1: zero free sides

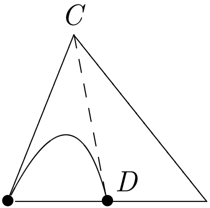
Zero free sides $\Rightarrow$ 7 or less good triangles.



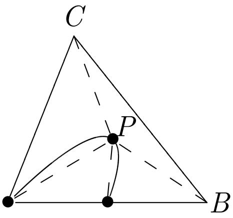
In other cases, cutting into good triangles is not so easy.

Situation: extra point needed

Sometimes we cannot do only with the original intersection points.

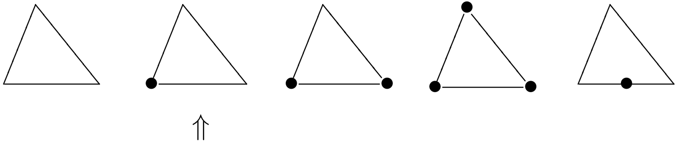


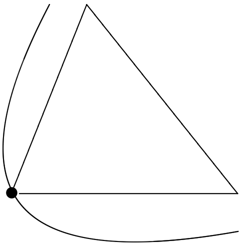
Lucky: CD is free



Unlucky: Find P such that PB,PC are free (fast)

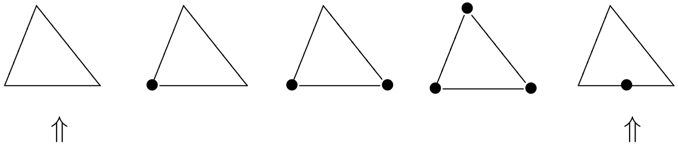
Integration on good triangles

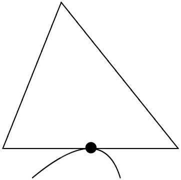




$$\iint_T \quad \text{or} \quad 0$$

Integration on good triangles





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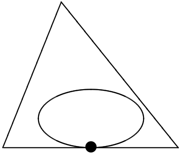
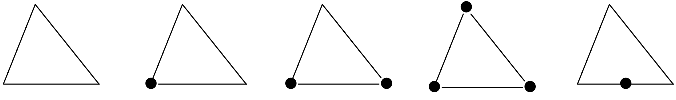


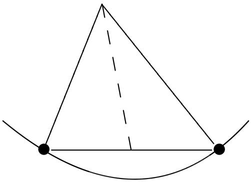
$$\iint_{ellipse} :$$

f $\rightarrow$circle + polar integration
or Green's theorem or ...

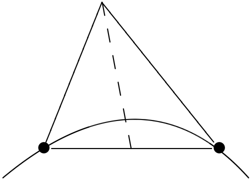
Integration on good triangles



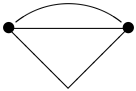
$\Uparrow$



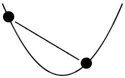
$$\iint_T \quad \text{or} \quad 0$$



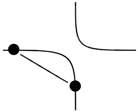
$\longrightarrow$



$\iint_{sector} - \iint_{triangle}$



$\iint dy\, dx$



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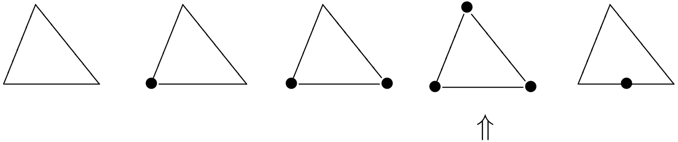
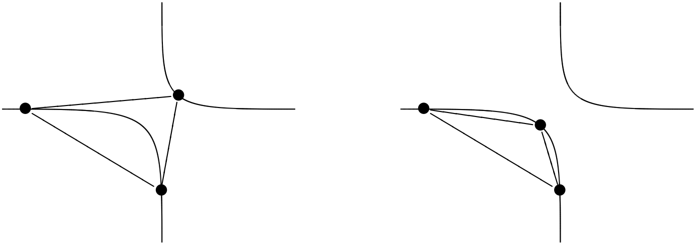
Integration on good triangles



- ellipse or parabola: $\iint_T$ or 0
- hyperbola: convert to $xy - 1$. In which half does each vertex lie? Easy.



Summary of the algorithm

1. Calculate intersection points with the three sides.
2. Cut in pieces.
    - Not many pieces.
    - Mostly, no new points are needed. When needed, easy to construct.
3. Integrate on each piece.

We have less cases, an easier way to discern them, and simpler formulas.

… in theory.

… in theory.

We are in the process of implementing our algorithm in MATLAB.

Practical challenges

Problem

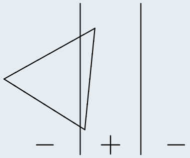
Scalings of very different orders of magnitude ruin the precision.

Solution

Use only rotations. Consequence: sizes are not normalized (they appear in the formulas as functions of the coefficients of f).

Problem

Integrals on small regions are calculated as differences of regions orders of magnitude larger. Precission is lost in the subtraction.



Solution

Find alternative formulas. Find a quick decision method to use the new, more complicated formulas when error hazard is detected.